Univerzitet u Zenici
Mašinski fakultet
Odsjek: Opšte mašinstvo

Sadržaj sveske sa vježbi iz predmeta Matematika 1

(akademska 2010/2011.)

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Algebarski izrazi

$(a+b)^2 = a^2+2ab+b^2$, $\quad (a+b)^2 = (a+b)(a+b)$

$(a-b)^2 = a^2-2ab+b^2$, $\quad (a-b)^2 = (a-b)(a+b)$

a^2+b^2

$a^2-b^2 = (a-b)(a+b)$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$(a-b)^3 = a^3-3a^2b+3ab^2-b^3$

$a^3+b^3 = (a+b)(a^2-ab+b^2)$

$a^3-b^3 = (a-b)(a^2+ab+b^2)$

① Uprostiti izraz: $\left(\frac{3}{a-1}+\frac{3a^2+3a+3}{1-a^2}:\frac{a^4-a}{a^3+1}\right)\cdot\frac{a-a^2}{3}$.

Rj.

$$\left(\frac{3}{a-1}+\frac{3a^2+3a+3}{1-a^2}:\frac{a^4-a}{a^3+1}\right)\cdot\frac{a-a^2}{3} =$$

$$=\left(\frac{3}{a-1}+\frac{3(a^2+a+1)}{-(a^2-1)}\cdot\frac{a^3+1}{a(a^3-1)}\right)\cdot\frac{-(a^2-a)}{3} =$$

$$=\left(\frac{3}{a-1}+\frac{3\cancel{(a^2+a+1)}}{-(a-1)\cancel{(a+1)}}\cdot\frac{\cancel{(a+1)}(a^2-a+1)}{a(a-1)\cancel{(a^2+a+1)}}\right)\cdot\frac{(-a)(a-1)}{3}$$

$$=\left(\frac{3}{a-1}+\frac{3(a^2-a+1)}{(-a)(a-1)^2}\right)\cdot\frac{(-a)(a-1)}{3} =$$

$$=\frac{3\cdot(-a)(a-1)+3(a^2-a+1)}{\cancel{(-a)}(a-1)^{\cancel{2}\,1}}\cdot\frac{\cancel{(-a)}\cancel{(a-1)}}{3} = \frac{3(-a^2+a+a^2-a+1)}{3(a-1)} = \frac{1}{a-1}$$

② Uprostiti izraz: $\left[\frac{1}{(b-a)^3}:\frac{1}{(a-b)^2}-\frac{1}{a+b}\right]:\frac{2a^2}{a^2-b^2}+1:(a^2+a)$;

Rj.

$$\left[\frac{1}{(b-a)^3}:\frac{1}{(a-b)^2}-\frac{1}{a+b}\right]:\frac{2a^2}{a^2-b^2}+1:(a^2+a) =$$

$$=\left[\frac{1}{((-1)(a-b))^3}\cdot\frac{(a-b)^2}{1}-\frac{1}{a+b}\right]\cdot\frac{a^2-b^2}{2a^2}+\frac{1}{a^2+a} =$$

$$=\left[\frac{(a-b)^2}{(-1)(a-b)^3}-\frac{1}{a+b}\right]\cdot\frac{a^2-b^2}{2a^2}+\frac{1}{a(a+1)}=$$

$$=\left[\frac{(-1)}{a-b}+\frac{(-1)}{a+b}\right]\cdot\frac{(a-b)(a+b)}{2a^2}+\frac{1}{a(a+1)}=\frac{-a-b-a+b}{(a-b)(a+b)}\cdot\frac{(a-b)(a+b)}{2a^2}+\frac{1}{a(a+1)}=$$

$$=\frac{-2a}{2a^2}+\frac{1}{a(a+1)}=\frac{(-1)}{a}+\frac{1}{a(a+1)}=\frac{-a-1+1}{a(a+1)}=\frac{-1}{a+1}$$

③ Uprostiti izraz: $\dfrac{(\sqrt{a}-\sqrt{b})^3+2\sqrt{a^3}+b\sqrt{b}}{a\sqrt{a}+b\sqrt{b}}+\dfrac{3\sqrt{ab}-3b}{a-b}$

Rj.

$$\frac{(\sqrt{a}-\sqrt{b})^3+2\sqrt{a^3}+b\sqrt{b}}{a\sqrt{a}+b\sqrt{b}}+\frac{3\sqrt{ab}-3b}{a-b}=$$

$$=\frac{\underline{\underline{\sqrt{a^3}}}-3\sqrt{a^2b}+3\sqrt{ab^2}\boxed{-\sqrt{b^3}}+\underline{\underline{2\sqrt{a^3}}}\boxed{+\sqrt{b^3}}}{\sqrt{a^3}+\sqrt{b^3}}+\frac{3\sqrt{ab}-3\sqrt{b^2}}{\sqrt{a^2}-\sqrt{b^2}}=$$

$$=\frac{3\sqrt{a^3}-3\sqrt{a^2b}+3\sqrt{ab^2}}{\sqrt{a^3}+\sqrt{b^3}}+\frac{3\sqrt{b}\,\cancel{(\sqrt{a}-\sqrt{b})}}{\cancel{(\sqrt{a}-\sqrt{b})}(\sqrt{a}+\sqrt{b})}=$$

$$=\frac{3\sqrt{a}\,\cancel{(\sqrt{a^2}-\sqrt{ab}+\sqrt{b^2})}}{(\sqrt{a}+\sqrt{b})\cancel{(\sqrt{a^2}-\sqrt{ab}+\sqrt{b^2})}}+\frac{3\sqrt{b}}{\sqrt{a}+\sqrt{b}}=\frac{3\sqrt{a}+3\sqrt{b}}{\sqrt{a}+\sqrt{b}}=\frac{3(\sqrt{a}+\sqrt{b})}{\sqrt{a}+\sqrt{b}}=3$$

④ Uprostiti izraz: $\left(\dfrac{m^{\frac{2}{3}}}{m^{\frac{2}{3}}-2m^{-\frac{1}{3}}}-\dfrac{m^{\frac{4}{3}}}{m^{\frac{4}{3}}-m^{\frac{1}{3}}}\right)(m-3+2m^{-1})-\left(\dfrac{2m-3}{m+5}\right)^0$.

$$\left(\frac{m^{\frac{2}{3}}}{m^{\frac{2}{3}}-2m^{-\frac{1}{3}}}-\frac{m^{\frac{4}{3}}}{m^{\frac{4}{3}}-m^{\frac{1}{3}}}\right)(m-3+2m^{-1})-\left(\frac{2m-3}{m+5}\right)^0=$$

$$=\left(\frac{\sqrt[3]{m^2}}{\sqrt[3]{m^2}-\frac{2}{\sqrt[3]{m}}}-\frac{\sqrt[3]{m^4}}{\sqrt[3]{m^4}-\sqrt[3]{m}}\right)\left(m-3+\frac{2}{m}\right)-1=$$

$$=\left(\frac{\sqrt[3]{m^2}}{\frac{\sqrt[3]{m^3}-2}{\sqrt[3]{m}}}-\frac{\sqrt[3]{m^3}\cdot\cancel{\sqrt[3]{m}}}{\cancel{\sqrt[3]{m}}(\sqrt[3]{m^3}-1)}\right)\frac{m^2-3m+2}{m}-1=$$

$$=\left(\frac{\sqrt[3]{m^3}}{\sqrt[3]{m^3}-2}-\frac{\sqrt[3]{m^3}}{\sqrt[3]{m^3}-1}\right)\cdot\frac{m^2-m-2m+2}{m}-1=\left(\frac{m}{m-2}-\frac{m}{m-1}\right)\cdot\frac{m(m-1)-2(m-1)}{m}-1=$$

$$= \frac{m(m-1)-m(m-2)}{\cancel{(m-2)}\cancel{(m-1)}} \cdot \frac{\cancel{(m-2)}\cancel{(m-1)}}{m} - 1 = \frac{\cancel{m}(m-1-m+2)}{\cancel{m}} - 1 = 1-1=0$$

⑤ Uprostiti izraz: $\frac{\sqrt{a}-\sqrt{x}}{\sqrt[4]{a}-\sqrt[4]{x}} - \left(\frac{a+\sqrt[4]{ax^3}}{\sqrt{a}+\sqrt[4]{ax}} - \sqrt[4]{ax}\right) : \left(\sqrt[4]{a}-\sqrt[4]{x}\right)$.

⑥ Uprostiti izraz: $\left[(a^{\frac{1}{2}}-b^{\frac{1}{2}})^{-1}(a^{\frac{3}{2}}-b^{\frac{3}{2}}) - \frac{1}{(\sqrt{a}+\sqrt{b})^{-2}}\right] : \sqrt[3]{ab\sqrt{ab}} + \frac{1}{1+\left[a(1-a^2)^{-\frac{1}{2}}\right]^2}$.

⑦ Uprostiti izraz: $\left(\frac{\sqrt[4]{a^3}-1}{\sqrt[4]{a}-1}+\sqrt[4]{a}\right)^{\frac{1}{2}} \cdot \left(\frac{\sqrt[4]{a^3}+1}{\sqrt[4]{a}+1}-\sqrt{a}\right) \cdot (a-\sqrt{a^3})^{-1}$.

Rješenja: 5. $2\sqrt[4]{x}$ 6. $-a^2$ 7. $\frac{1}{a}$

Kvadratne jednačine i kvadratna f-ja

Jednačina oblika $ax^2+bx+c=0$ $(a,b,c\in\mathbb{R},\ a\neq 0)$ zove se kvadratna jednačina.

F-ja $f:\mathbb{R}\to\mathbb{R}$ gdje je $f(x)=ax^2+bx+c$ (drugačije napisano $y=ax^2+bx+c$) $a,b,c\in\mathbb{R}$, $a\neq 0$ zove se kvadratna f-ja ili polinom drugog stepena.

① Riješiti kvadratne jednačine:

a) $(2x-3)^2=15$ b) $4x^2+9=0$ c) $5x^2-7x=0$.

Rj. a) $(2x-3)^2=15$

$$2x-3=\pm\sqrt{15}$$
$$2x=\pm\sqrt{15}+3$$
$$x=\pm\frac{\sqrt{15}}{2}+\frac{3}{2}$$
$$x_1=-\frac{\sqrt{15}}{2}+\frac{3}{2}$$
$$x_2=\frac{\sqrt{15}}{2}+\frac{3}{2}$$

b) $4x^2+9=0$

$$4x^2=-9$$
$$x^2=-\frac{9}{4}$$
$$x=\pm\sqrt{\frac{-9}{4}}$$
$$x=\pm\sqrt{\frac{9}{4}i^2}$$
$$x_1=-\frac{3}{2}i$$
$$x_2=\frac{3}{2}i$$

c) $5x^2-7x=0$

$$(5x-7)x=0$$

$5x-7=0$ ili $x=0$

$$5x=7$$
$$x=\frac{7}{5}$$

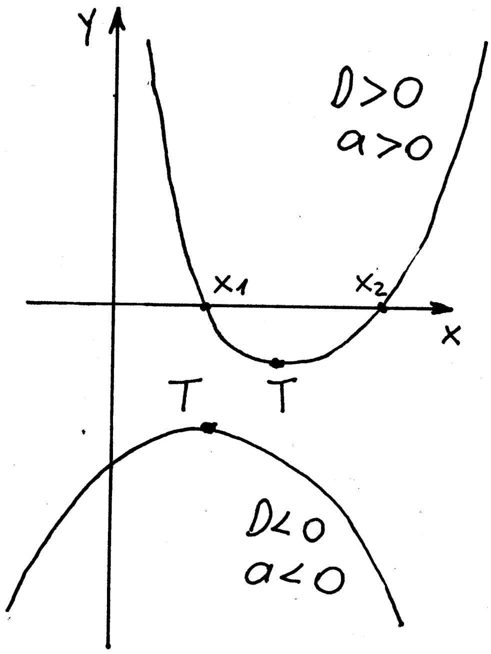
Rješenje kvadratne jednačine je $x=\frac{7}{5}$ ili $x=0$.

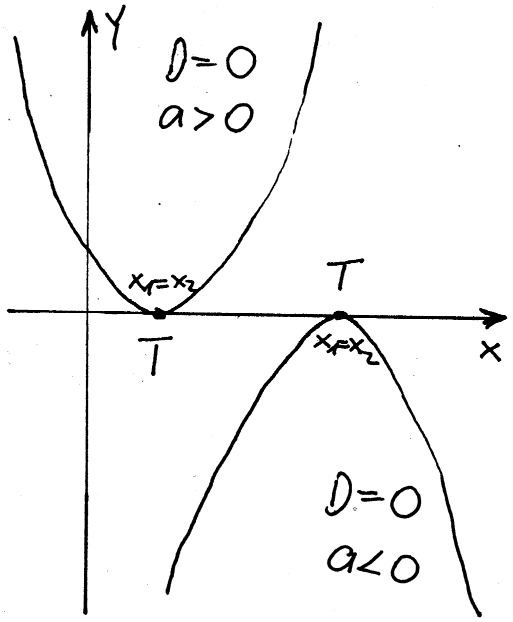
$D=b^2-4ac$, D diskriminanta

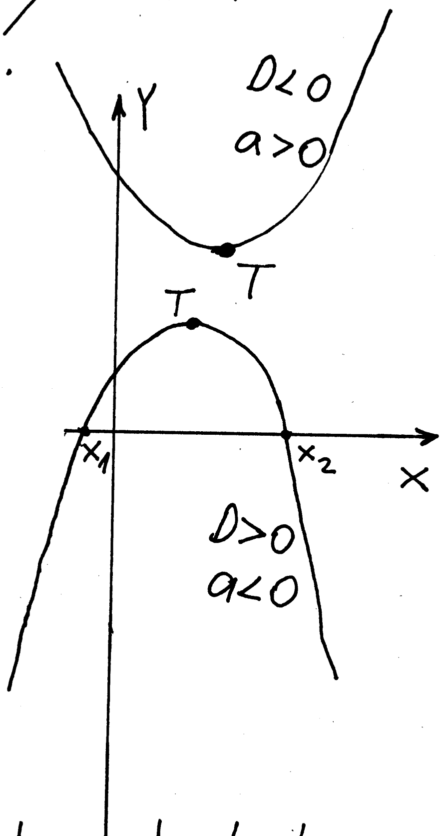
Grafik kvadratne f-je $f(x)=ax^2+bx+c$ ima oblik parabole koja ima nule u tačkama $x_{1,2}=\frac{-b\pm\sqrt{D}}{2a}$.

Ako je $a>0$ minimum f-je je u tački $T(-\frac{b}{2a}, -\frac{D}{4a})$.

Ako je $a<0$ kvadratna f-ja ima maksimum u tački $T(-\frac{b}{2a}, -\frac{D}{4a})$ (u istoj tački).







Primjetimo da:

$$D=b^2-4ac=\begin{cases} >0, & x_1\neq x_2 \text{ realni različiti brojevi}\\ =0, & x_1=x_2 \text{ realni brojevi}\\ <0, & x_1, x_2 \text{ konjugovano kompleksni brojevi}\end{cases}$$

(2.) Grafički predstaviti i naći ekstrem f-je $y=x^2-6x+8$.

Rj. Tražimo nule f-je (u kojim tačkama f-ja siječe x-osu).

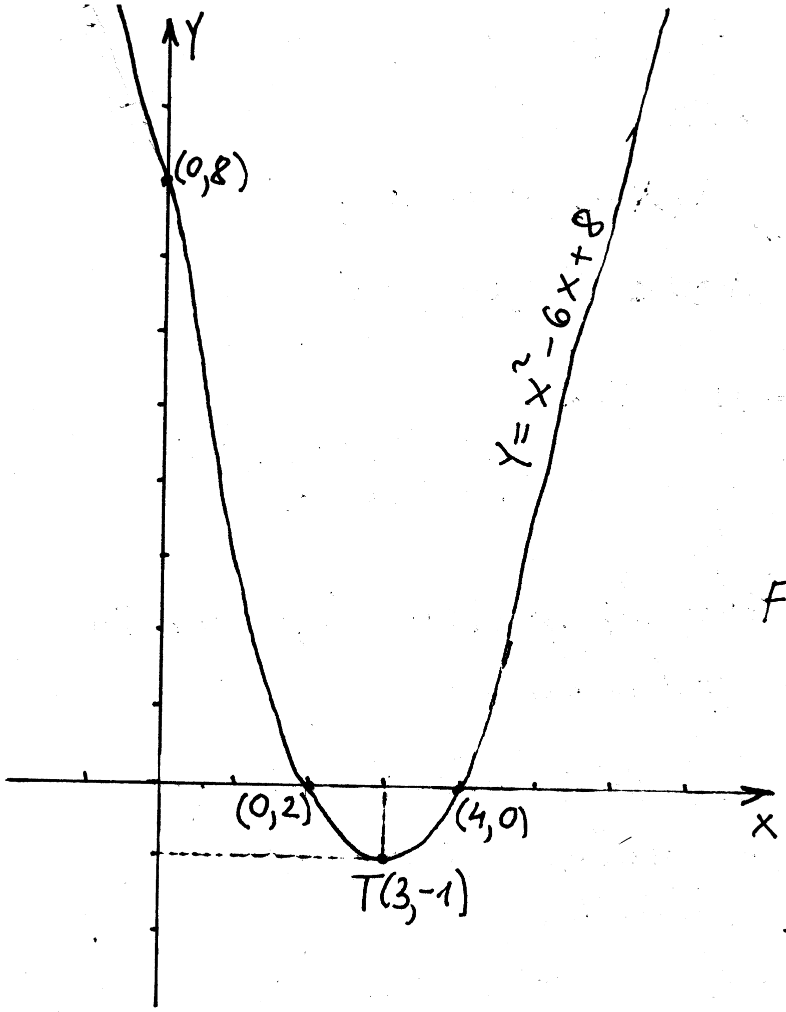
$x^2-6x+8=0$

$D=36-32=4$

$x_{1,2}=\frac{6\pm2}{2}$

$x_1=\frac{4}{2}=2 \quad x_2=\frac{8}{2}=4$

Nule f-je su $x_1=2$; $x_2=4$



Tražimo presjek sa Y-osom:

$$f(x)=x^2-6x+8$$

$$f(0)=8$$

(0,8) je tačka presjeka grafa f-je sa Y-osom.

Tražimo ekstreme f-je

$a=1>0 \Rightarrow$ f-ja je oblika $\cup$

F-ja ima minimum u tački $T\left(-\frac{b}{2a},-\frac{D}{4a}\right)$

$$-\frac{b}{2a}=-\frac{(-6)}{2}=3$$

$$-\frac{D}{4a}=-\frac{4}{4}=-1 \qquad T(3,-1)$$

Jednačinu $ax^2+bx+c=0$ možemo rastaviti na faktore pomoću formule $a(x-x_1)(x-x_2)=0$. $\left(x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)$.

③ Sljedeće jednačine rastaviti na faktore:

a) $3x^2+5x-8=0$ b) $2x^2+13x-7=0$ c) $6x^2-x-2=0$.

Rj. a) $3x^2+5x-8=0$

$D=25+96=121$

$x_{1,2}=\frac{-5\pm11}{6}$

$x_1=\frac{-16}{6}=-\frac{8}{3}$

$x_2=\frac{6}{6}=1$

$3(x+\frac{8}{3})(x-1)=0$

Jednačina rastavljena na faktore je

$(3x+8)(x-1)=0$

b) $2x^2+13x-7=0$

$D=169+56$

$x_{1,2}=\frac{-13\pm15}{4}$

$x_1=\frac{-28}{4}=-7$

$x_2=\frac{2}{4}=\frac{1}{2}$

$2(x+7)(x-\frac{1}{2})=0$

Jednačina rastavljena na faktore je

$(x+7)(2x-1)=0$

c) $6x^2-x-2=0$

$D=1+48=49$

$x_{1,2}=\frac{1\pm7}{12}$

$x_1=\frac{-6}{12}=-\frac{1}{2}$

$x_2=\frac{8}{12}=\frac{2}{3}$

$6(x+\frac{1}{2})(x-\frac{2}{3})=$

$2\cdot3\cdot(x+\frac{1}{2})(x-\frac{2}{3})=$

$=2(x+\frac{1}{2})\cdot3(x-\frac{2}{3})=$

$=(2x+1)(3x-2)$

④ Za koju vrijednost parametra λ jednačina $\lambda^2(x-1)=4\lambda(x-2)+16$ ima više od jednog rješenja.

Rj. $\lambda^2(x-1)=4\lambda(x-2)+16$

$\underline{\lambda^2 x}-\lambda^2=\underline{4\lambda x}-8\lambda+16$

$\lambda^2 x-4\lambda x=\lambda^2-8\lambda+16$

$\nearrow$ $\lambda(\lambda-4)x=(\lambda-4)^2$

$\lambda=0:$ $\quad 0\cdot x=16$

nema rješenja

$\lambda=4:$ $\quad 0\cdot x=0$

∞ mnogo rješenja

Za $\lambda=4$ jednačina ima ∞ mnogo rješenja.

⑤ Odrediti parametar λ tako da rješenja jednačine $8(x^2-1)=(\lambda-2)x-\lambda$ budu jednaka.

Rj. $8(x^2-1)=(\lambda-2)x-\lambda$

$8x^2-8-(\lambda-2)x+\lambda=0$

$8x^2+(-\lambda+2)x+\lambda-8=0$

$D=(-\lambda+2)^2-4\cdot 8\cdot(\lambda-8)$

$=\lambda^2-4\lambda+4-32\lambda+256$

$=\lambda^2-36\lambda+260$

Za $D=0$ rješenja svake kvadratne jednačine su jednaka.

$\lambda^2-36\lambda+260=0$

$D=1296-1040=256$

$\lambda_{1,2}=\frac{36\pm16}{2}$

$\lambda_1=\frac{20}{2}=10$ $\qquad \lambda_2=\frac{52}{2}=26$

Rješenja jednačine će biti jednaka za $\lambda=10$ ili za $\lambda=26$.

⑥ Grafički predstaviti i naći ekstrem f-je

a) $y=-\frac{1}{2}x^2+x+1\frac{1}{2}$

b) $y=2x^2+9x-5$

⑦ Rastaviti na faktore

a) $6x^2+5bx+b^2=0$

b) $8x^2+2px-3p^2=0$

⑧ Za koje vrijednosti parametra λ su rješenja jednačine $(1-\lambda)x^2-2(1+\lambda)x+3(1+\lambda)=0$ realna i različita.

Rješenja:

6. a) $T(1,2)$ b) $T(-2\frac{1}{4},-15\frac{1}{8})$

7. a) $(2x+b)(3x+b)=0$

b) $(4x+3p)(2x-p)=0$

8. $\lambda\in(-\infty,-1)\cup(\frac{1}{2},+\infty)$

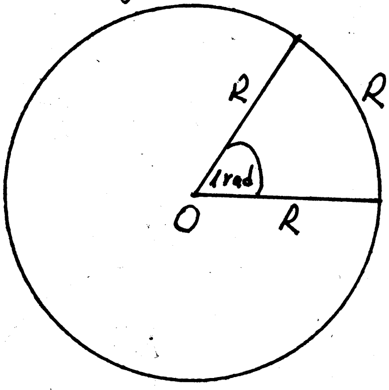
Trigonometrija

Najpoznatije jedinice za mjerenje veličine ugla su radijan i stepen.

2π rad $= 360°$ $\frac{\pi}{2}$ rad $= 90°$ $\frac{\pi}{3}$ rad $= 60°$

π rad $= 180°$ $\frac{\pi}{4}$ rad $= 45°$ $\frac{\pi}{6}$ rad $= 30°$

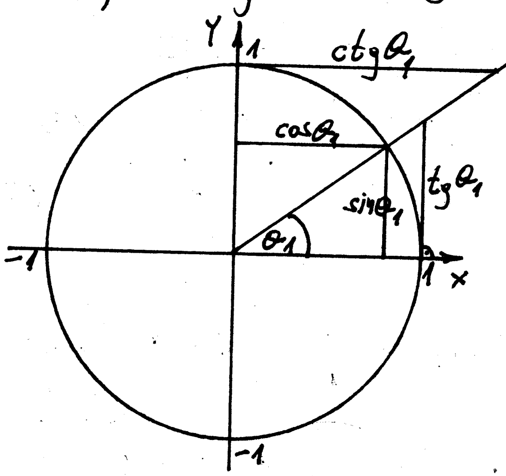
Stepen je devedeseti dio pravog ugla.

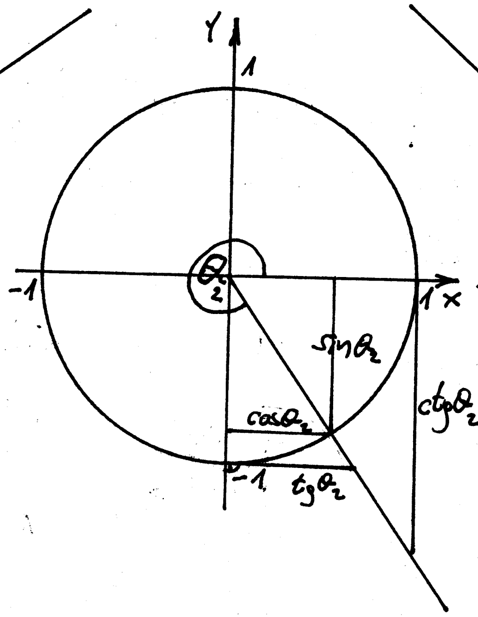


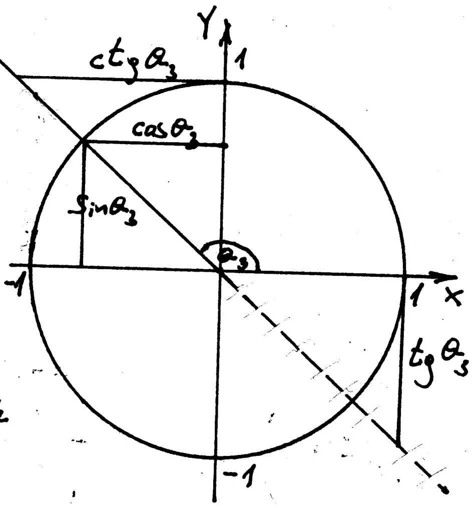
Radijan je veličina centralnog ugla nad lukom (kružnice) čija je dužina jednaka poluprečniku (slika).

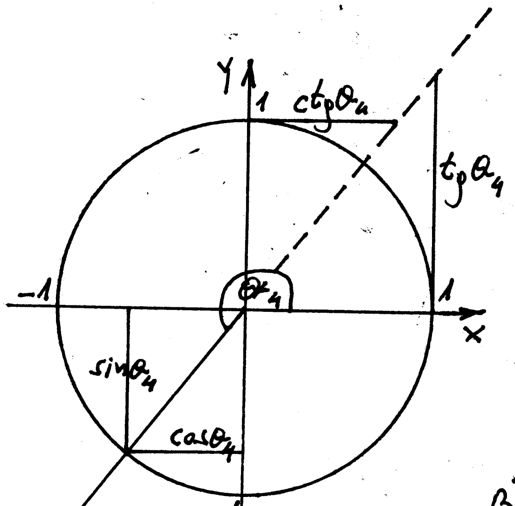
$$1 \text{ rad} = \frac{360°}{2\pi} \qquad 1 \text{rad} \approx 57°17'44''$$

Krug sa centrom u koordinatnom početku poluprečnika 1 (jedan) nam pomaže da definišemo sinus (sin), kosinus (cos), tangens (tg) i kotangens (ctg) proizvoljnog ugla.

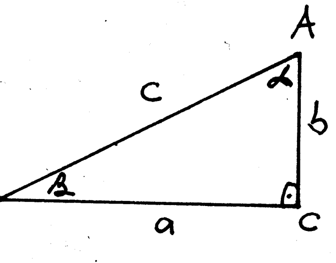








Ako je dat pravougli trougao:

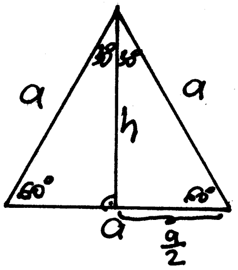


$$\sin\alpha = \frac{a}{c}, \quad \sin\beta = \frac{b}{c},$$

$$\cos\alpha = \frac{b}{c}, \quad \cos\beta = \frac{a}{c}$$

$$tg\,\alpha = \frac{\sin\alpha}{\cos\alpha} \quad ctg\,\alpha = \frac{\cos\alpha}{\sin\alpha}$$

$$\sin^2\alpha + \cos^2\alpha = 1$$



$$\sin 60° = \frac{h}{a}$$

$$h^2 = a^2 - \frac{a^2}{4}$$

$$h = \frac{a\sqrt{3}}{2}$$

$$\sin 60° = \frac{\frac{a\sqrt{3}}{2}}{2} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{\frac{a}{2}}{a} = \frac{1}{2}$$

$$\sin 30° = \frac{\frac{a}{2}}{a} = \frac{1}{2}$$

$$\cos 30° = \frac{h}{a} = \frac{\frac{a\sqrt{3}}{2}}{a} = \frac{\sqrt{3}}{2}$$

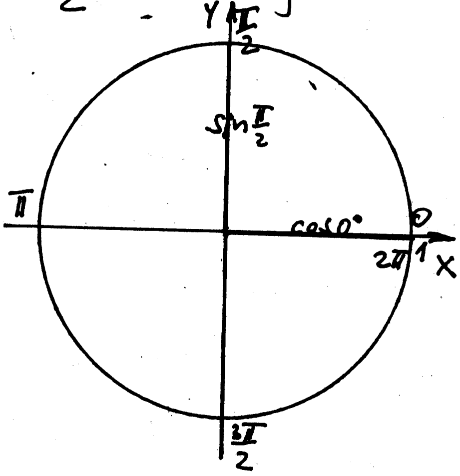
α	30°	60°	45°
$\sin\alpha$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$
$\cos\alpha$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$
$tg\,\alpha$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	1
$ctg\,\alpha$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	1

① Izračunati: a) $\cos 0°$ b) $\sin\frac{\pi}{2}$ c) $tg\frac{3\pi}{2}$ d) $ctg\,\pi$

e) $\sin 2\pi$ f) $ctg\frac{\pi}{2}$ g) $\cos\frac{3\pi}{2}$ h) $tg\,\pi$ i) $\cos\frac{\pi}{2}$

j)$^\vee$ $\sin\pi$ k)$^\vee$ $tg\,0°$ l)$^\vee$ $ctg\frac{3\pi}{2}$ m)$^\vee$ $\sin\frac{3\pi}{2}$ n)$^\vee$ $ctg\,2\pi$

o)$^\vee$ $\cos\pi$ p)$^\vee$ $tg\frac{\pi}{2}$



Rješenje:

a) $\cos 0° = 1$ b) $\sin\frac{\pi}{2} = 1$ c) $tg\frac{3\pi}{2} = -\infty$ d) $ctg\,\pi = -\infty$

e) $\sin 2\pi = 0$ f) $ctg\frac{\pi}{2} = 0$ g) $\cos\frac{3\pi}{2} = 0$ h) $tg\,\pi = 0$

i) $\cos\frac{\pi}{2} = 0$

② Izračunati:

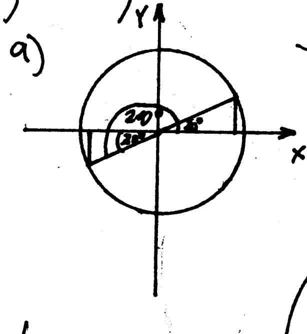
a) $\sin 210°$ b) $\cos 120°$ c) $\sin 330°$ d) $\cos 240°$ e) $\sin 150°$

f)$^\vee$ $\cos 300°$ g)$^\vee$ $\sin 240°$ h)$^\vee$ $\cos 330°$ i)$^\vee$ $\sin 300°$ k)$^\vee$ $\cos 150°$

l)$^\vee$ $\sin 120°$ m)$^\vee$ $\cos 210°$

Rješenje:

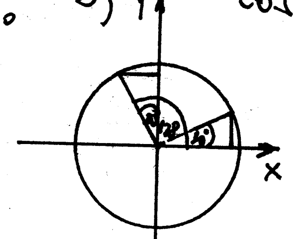
a)



$\sin 210° = -\sin 30°$

$= -\frac{1}{2}$

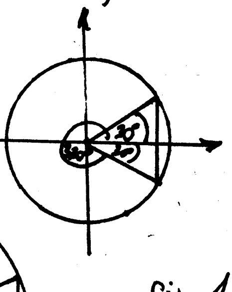
b)



$\cos 120° = -\sin 30°$

$= -\frac{1}{2}$

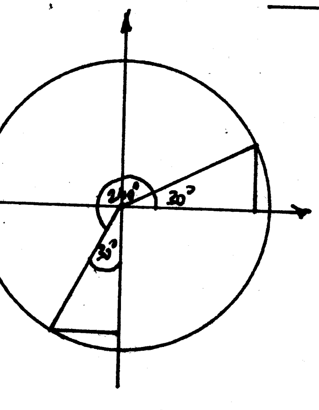
c)



$\sin 330° = -\sin 30°$

$= -\frac{1}{2}$

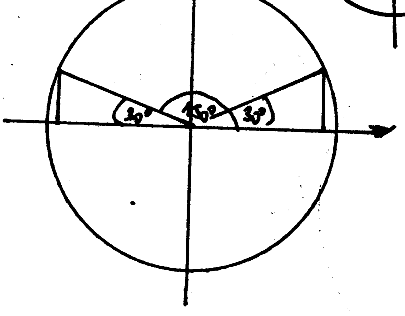
d)



$\cos 240° =$

$= -\sin 30° = -\frac{1}{2}$

e)



$\sin 150° =$

$= \sin 30° = \frac{1}{2}$.

③$^\vee$ Pojednostaviti zadane izraze:

a) $\frac{1}{\sin^2\alpha} - 1$ b) $\frac{1-\cos^2\alpha}{\sin\alpha\cos\alpha}$ c) $\frac{1+\cos^2\alpha - \sin^2\alpha}{\sin^2\alpha}$ d) $\frac{1+\sin\alpha - \cos^2\alpha}{1+\sin\alpha}$

Matematička indukcija

Matematička tvrdnja je tačna (istinita) za svaki prirodan broj ($n \in \mathbb{N}$) ako su ispunjena sljedeća dva uslova:

a) BAZA INDUKCIJE

Tvrdnja je tačna za broj 1.

b) INDUKCISKI KORAK

Ako na osnovu pretpostavke da je tvrdnja tačna za $k \leq n$ $(k=1,2,\dots,n)$ slijedi da je istinita i za broj $n+1$.

(#) Matematičkom indukcijom dokazati da za sve prirodne brojeve važe sljedeće jednakosti:

a) $1+3+5+\dots+(2n-1)=n^2$

b) $1^3+2^3+3^3+\dots+n^3=\left[\frac{n(n+1)}{2}\right]^2$

c) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots\frac{1}{n(n+1)}=\frac{n}{n+1}$

Rj. a) $1+3+5+\dots+(2k-1)=k^2$

BAZA INDUKCIJE

Pokažimo da je tvrdnja tačna za $k=1$. $1=1^2$ Tvrdnja je tačna za $k=1$.

KORAK INDUKCIJE

Pretpostavimo da je tvrdnja tačna za $k=1,2,\dots,n$ tj. $1+3+5+\dots+(2k-1)=k^2$ za sve k od 1 do n. Pokažimo da je tvrdnja tačna za $n+1$.

tj. pokažimo da $1+3+\dots+(2n+1)=(n+1)^2$

$$\underline{\underline{1+3+5+\dots+(2n-1)}}+(2n+1)\xrightarrow[]{\text{prema pretpostavci}}\underline{\underline{n^2}}+(2n+1)=n^2+2n+1=(n+1)^2$$

Dobili smo $1+3+5+\dots+(2n+1)=(n+1)^2$ što je i trebalo.

ZAKLJUČAK

Jednakost $1+3+\dots+(2n-1)=n^2$ je tačna za sve prirodne brojeve.

b) $1^3+2^3+3^3+\dots+k^3=\left[\frac{k(k+1)}{2}\right]^2$

BAZA INDUKCIJE

Pokažimo da je tvrdnja tačna za $k=1$. $1^3=\left(\frac{1(1+1)}{2}\right)^2=1^2$ Tvrdnja je tačna za $k=1$

KORAK INDUKCIJE

Pretpostavimo da je $1^3+2^3+3^3+\dots+k^3=\left[\frac{k(k+1)}{2}\right]^2$ za $\forall k=1,2,\dots,n$

Na osnovu ove pretpostavke pokažimo da $1^3+2^3+\dots+(n+1)^3=\left(\frac{(n+1)(n+2)}{2}\right)^2$.

Imamo $1^2+2^3+\dots+n^2+(n+1)^3 \overset{\text{na osnovu pretpostavke}}{=} \left(\frac{n(n+1)}{2}\right)^2+(n+1)^3=\frac{n^2(n+1)^2}{4}+\frac{4(n+1)^3}{4}=$

$=\frac{(n+1)^2(n^2+4(n+1))}{4}=\frac{(n+1)^2(n^2+4n+4)}{4}=\frac{(n+1)^2(n+2)^2}{4}=\left(\frac{(n+1)(n+2)}{2}\right)^2$ što je i trebalo dobiti

ZAKLJUČAK

Jednakost je tačna za sve prirodne brojeve.

c) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{k(k+1)}=\frac{k}{k+1}$

BAZA INDUKCIJE

... sami ...

KORAK INDUKCIJE

$\dots \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}=$

$\overset{\text{na osnovu pretpostavke}}{=} \frac{n}{(n+1)}+\frac{1}{(n+1)(n+2)}=\frac{n(n+2)+1}{(n+1)(n+2)}=\frac{n^2+2n+1}{(n+1)(n+2)}=\frac{(n+1)^2}{(n+1)(n+2)}=\frac{n+1}{n+2}$ što je i trebalo dobiti

ZAKLJUČAK

Jednakost je tačna za sve prirodne brojeve

① Dokázati da je $2^n \geq 2n$ za $\forall (n\in \mathbb{N})$.

Rj. $2^k \geq 2\cdot k$, k prirodan broj

BAZA INDUKCIJE

$k=1$: $2^1 \geq 2\cdot 1$ tj. $2\geq 2$ tačno

Za $k=1$ tvrdnja je tačna.

INDUKCISKI KORAK

Pretpostavimo da je $2^k \geq 2k$ za svaki $k=1,2,\dots,n$.

Na osnovu toga, dokažimo da je tačno i $2^{n+1}\geq 2(n+1)$.

$$\underline{\underline{2^{n+1}}}=2^n\cdot 2=2^n+2^n\geq 2^n+2 \overset{\text{na osnovu pretpostavke}}{\geq} 2n+2=\underline{\underline{2(n+1)}}$$

tj. $2^{n+1}\geq 2(n+1)$ što je i trebalo pokazati.

ZAKLJUČAK

Nejednakost $2^n\geq 2n$ je tačna za svaki prirodan broj.

② Dokazati da je nejednakost $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}\geq\sqrt{n}$ tačna za svaki prirodan broj.

Rj. $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{k}}\geq\sqrt{k}$, $k=1,2,3,\dots$

BAZA INDUKCIJE $k=1$: $\frac{1}{\sqrt{1}}\geq\sqrt{1}$ tj. $1\geq1$ tačno. Za $k=1$ nejednakost je tačna.

INDUKCISKI KORAK

Pretpostavimo da je nejednakost $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{k}}\geq\sqrt{k}$ tačna za svaki $k=1,2,\dots,n$.

Na osnovu ove pretpostavke dokažimo da je

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n+1}}\geq\sqrt{n+1}.$$

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}\overset{\text{prema pretpostavci}}{\geq}\sqrt{n}+\frac{1}{\sqrt{n+1}}=\frac{\sqrt{n}\cdot\sqrt{n+1}+1}{\sqrt{n+1}}$$

$$=\frac{\sqrt{n^2+n}+1}{\sqrt{n+1}}>\frac{n+1}{\sqrt{n+1}}\cdot\frac{\sqrt{n+1}}{\sqrt{n+1}}=\sqrt{n+1}$$ što je i trebalo dobiti

ZAKLJUČAK

Nejednakost je tačna za svaki prirodan broj.

③ Metodom matematičke indukcije dokazati da je 5^n+2^{n+1} djeljiv sa 3 za svaki prirodan broj n.

Rj. Treba dokazati da je broj 5^k+2^{k+1} djeljiv sa 3 za $\forall k\in\mathbb{N}$.

BAZA INDUKCIJE

$k=1$: $5^1+2^{1+1}=5+2^2=5+4=9$ 9 je djeljiv sa 3.

Za $k=1$ tvrdnja je tačna.

INDUKCISKI KORAK

Pretpostavimo da je 5^k+2^{k+1} djeljivo sa 3 za $k=1,2,\dots,n$.

Na osnovu ove pretpostavke dokažimo da je i $5^{n+1}+2^{n+1+1}$ djeljivo sa 3.

$$5^{n+1}+2^{n+1+1}=5^{n+1}+2^{n+2}=5\cdot5^n+2\cdot2^{n+1}=\underline{2\cdot5^n}+2\cdot2^{n+1}\underline{+3\cdot5^n}$$

+

$$=\underbrace{2(5^n+2^{n+1})}_{\text{ovaj dio je prema pretpostavci djeljiv sa 3}}+\underbrace{3\cdot5^n}_{\text{ovo je djeljivo sa 3}}$$

Prema tome $5^{n+1}+2^{n+2}$ je djeljivo sa 3.

ZAKLJUČAK

5^k+2^{k+1} je djeljivo sa 3 za svaki prirodan broj k.

④ Metodom matematičke indukcije dokazati da jednakost $1^3+2^3+3^3+\ldots+n^3=\frac{n^2(n+1)^2}{4}$ vrijedi za sve prirodne brojeve.

Rj. $1^3+2^3+\ldots+k^3=\frac{k^2(k+1)^2}{4}$, k je prirodan broj..

BAZA INDUKCIJE

$k=1$: $1^3=\frac{1^2\cdot(1+1)^2}{4} \Rightarrow 1=\frac{4}{4} \Rightarrow 1=1$ što je tačno.

Za $k=1$ jednakost je tačna

INDUKCISKI KORAK

Pretpostavimo da je $1^3+2^3+\ldots+k^3=\frac{k^2(k+1)^2}{4}$ tačno za $k=1,\ldots,n$

Na osnovu ove pretpostavke dokažimo da je

$$1^3+2^3+\ldots+(n+1)^3=\frac{(n+1)^2(n+2)^2}{4}.$$

$$1^3+2^3+\ldots+n^3+(n+1)^3 \overset{\text{prema pretpostavci}}{=} \frac{n^2(n+1)^2}{4}+(n+1)^3=$$

$$=\frac{n^2(n+1)^2+4(n+1)^3}{4}=\frac{(n+1)^2(n^2+4n+4}{4}=\frac{(n+1)^2(n+2)^2}{4}$$ što je i trebalo dobiti

ZAKLJUČAK

Jednakost je tačna za sve prirodne brojeve.

⑤ Dokazati da je $4^n+15n-1$ djeljivo sa 9 za svaki prirodan broj n.

Rj. Treba dokazati da je $4^k+15k-1$ djeljivo sa 9 za $\forall(k\in N)$.

BAZA INDUKCIJE

$k=1$: $4^1+15\cdot1-1=4+15-1=18$, 18 je djeljivo sa 9.

Tvrdnja je tačna za $k=1$.

INDUKCISKI KORAK

Pretpostavimo da je $4^k+15\cdot k-1$ djeljivo sa 9 za $k=1,2,\ldots,n$.

Na osnovu ove pretpostavke dokažimo da je $4^{n+1}+15(n+1)-1$ tj. $4^{n+1}+15n+14$ djeljivo sa 9.

$$4^{n+1}+15n+14 = 4\cdot 4^n+15n-1+15 = 4\cdot 4^n+2\cdot 15n-2+16-15n =$$

$$= 4\cdot 4^n+4\cdot 15n-4+18-3\cdot 15n = 4(4^n+15n-1)+18-9\cdot 5n$$

$$= 4\underbrace{(4^n+15n-1)}_{\text{ovo je prema pretpostavci djeljivo sa 9}}+\underbrace{9(2-5n)}_{\text{ovo je djeljivo sa 9}}$$

Prema tome $4^{n+1}+15n+14$ je djeljivo sa 9.

ZAKLJUČAK

$4^n+15n-1$ je djeljivo sa 9 za svaki prirodan broj n.

⑥ Dokazati Bernulijevu nejednakost $(1+h)^n \geq 1+n\cdot h$ gdje je $h>-1$, a n pozitivan cijeli broj.

Rj. $(1+h)^k \geq 1+k\cdot h$, $h\in\mathbb{R}$, $h>-1$, $k\in\mathbb{N}$.

BAZA INDUKCIJE

$k=1$: $(1+h)^1 \geq 1+1\cdot h \Rightarrow 1+h \geq 1+h$ ovo je tačno

Za $k=1$ nejednakost je tačna.

INDUKCISKI KORAK

Pretpostavimo da je $(1+h)^k \geq 1+k\cdot h$ za $k=1,2,\dots,n$, $h>-1$.

Na osnovu ove pretpostavke dokažimo da je

$$(1+h)^{n+1} \geq 1+(n+1)h$$

$$(1+h)^{n+1} = (1+h)^n(1+h) \overset{\text{na osnovu pretpostavke}}{\geq} (1+nh)(1+h) = 1+h+nh+nh^2 \overset{h^2\geq 0}{\geq}$$

$$1+h+nh = 1+(n+1)h$$ što je i trebalo dobiti.

ZAKLJUČAK

Nejednakost je tačna za sve prirodne brojeve.

⑦ Metodom matematičke indukcije dokazati da jednakost $1\cdot 2+2\cdot 5+3\cdot 8+\dots+n(3n-1)=n^2(n+1)$ vrijedi za sve prirodne brojeve n.

⑧ Fibonačijev niz 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... je definisan rekurzivnom formulom $a_{n+1}=a_n+a_{n-1}$ gdje su $a_1=a_2=1$. Dokazati da je $NZD(a_n, a_{n+1})=1$ za sve privodne brojeve n (NZD je skraćenica od najveći zajednički djelilac, npr. $NZD(14,35)=7$).

Rj. $a_1=a_2=1$

$a_{k+1}=a_k+a_{k-1}$, $k\in\mathbb{N}$, $k\geq 2$

Treba dokazati da je $NZD(a_k, a_{k+1})=1$, za $\forall k\in\mathbb{N}$

BAZA INDUKCIJE

$k=1$: $a_1=1$, $a_2=1$, $NZD(a_1,a_2)=NZD(1,1)=1$ Tvrdnja je tačna za $k=1$.

INDUKCISKI KORAK

Pretpostavimo da je $NZD(a_k, a_{k+1})=1$ za sve $k=1,2,\dots,n$.
Na osnovu ove pretpostavke dokažimo da je $NZD(a_{n+1}, a_{n+2})=1$.

$a_{n+1}=a_n+a_{n-1}$

$a_{n+2}=a_{n+1}+a_n$

Označimo sa d NZD od brojeva a_{n+1} i a_{n+2} tj. $NZD(a_{n+1}, a_{n+2})=d$.

Nađimo, čemu je d jednako? Odredimo d.

$NZD(a_{n+1}, a_{n+2})=d \Rightarrow d\,|\,a_{n+1}$ (d djeli a_{n+1}) i $d\,|\,a_{n+2}$ (d djeli a_{n+2})

$$a_{n+2}=a_{n+1}+a_n \Rightarrow \left.\begin{array}{l} a_n=a_{n+2}-a_{n+1} \\ d\,|\,a_{n+1} \\ d\,|\,a_{n+2} \end{array}\right\} \Rightarrow d\,|\,a_n \text{ (d djeli } a_n)$$

Prema pretpostavci
$$\left.\begin{array}{l} d\,|\,a_n \\ d\,|\,a_{n+1} \\ \text{i } NZD(a_n,a_{n+1})=1 \end{array}\right\} \Rightarrow d=1$$
što je i trebalo dobiti

ZAKLJUČAK

$NZD(a_n,a_{n+1})=1$ za sve privodne brojeve n, $\{a_n\}$ Fibon. niz

⑨ Dokazati da je broj $2^{2n}-3n-1$ djeljiv sa 9 za svaki privodan broj veći od 1.

⑩ Metodom matematičke indukcije dokazati da jednakost
$$1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+\dots+n(n+1)(n+2)=\frac{n(n+1)(n+2)(n+3)}{4}$$
vrijedi za sve privodne brojeve n.

(11.) Dokazati Moavrov obrazac $(\cos x+i\sin x)^n=\cos nx+i\sin nx.$

Rj. $(\cos x+i\sin x)^k=\cos kx+i\sin kx,\quad k\in\mathbb{N}$

BAZA INDUKCIJE

k=1: $(\cos x+i\sin x)^1=\cos x+i\sin x,$ Za k=1, tvrdnja je tačna.

INDUKCISKI KORAK

Pretpostavimo da je $(\cos x+i\sin x)^k=\cos kx+i\sin kx$ za $k=1,2,\dots,n$

Na osnovu ove pretpostavke dokažimo da je

$$(\cos x+i\sin x)^{n+1}=\cos(n+1)x+i\sin(n+1)x.$$

$$(\cos x+i\sin x)^{n+1}=(\cos x+i\sin x)^n\cdot(\cos x+i\sin x)\quad \underline{\underline{\text{na osnovu pretpostavke}}}$$

$$=(\cos nx+i\sin nx)\cdot(\cos x+i\sin x)=\underline{\underline{\cos nx\cdot\cos x}}+i\cos nx\sin x$$
$$+i\sin nx\cos x+i^2\,\underline{\underline{\sin nx\sin x}}\overset{(*)}{=}$$

Adicione teoreme

$$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$$
$$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$$

$$\overset{(*)}{=}\cos(nx+x)+i\sin(nx+x)=\cos(n+1)x+i\sin(n+1)x$$

što je i trebalo dobiti

ZAKLJUČAK

Jednakost je tačna za sve prirodne brojeve.

(12.) Metodom matematičke indukcije dokazati da za svaki prirodan broj n vrijedi jednakost

$$1+q+q^2+\dots+q^n=\frac{1-q^{n+1}}{1-q}\quad \text{gdje je } q\in\mathbb{R}\setminus\{1\}.$$

Rj. $1+q+q^2+\dots+q^k=\frac{1-q^{k+1}}{1-q},\quad q\in\mathbb{R}\setminus\{1\},\ k\in\mathbb{N}$

BAZA INDUKCIJE

k=1: $1+q=\frac{1-q^{1+1}}{1-q}=\frac{1-q^2}{1-q}=\frac{(1-q)(1+q)}{(1-q)}$ tj. $1+q=1+q$

Za k=1 jednakost je tačna

INDUKCISKI KORAK

Pretpostavimo da je $1+q+q^2+\dots+q^k=\frac{1-q^{k+1}}{1-q}$ za $k=1,2,\dots,n$.

Na osnovu ove pretpostavke dokažimo da je

$$1+q+q^2+\dots+q^{n+1}=\frac{1-q^{n+2}}{1-q}.$$

$$1+q+q^2+\dots+q^n+q^{n+1}\overset{\text{prema pretpostavci}}{=\!=}\frac{1-q^{n+1}}{1-q}+q^{n+1}=\frac{1-q^{n+1}+q^{n+1}(1-q)}{1-q}$$

$$=\frac{1-q^{n+1}+q^{n+1}-q^{n+1+1}}{1-q}=\frac{1-q^{n+2}}{1-q}$$ što je i trebalo dobiti

ZAKLJUČAK

Jednakost je tačna za sve prirodne brojeve.

(13.) Ako su $x_1, x_2, \dots, x_n$ nenegativni realni brojevi, onda aritmetičku sredinu (prosjek) definišemo kao broj $A=\frac{1}{n}(x_1+x_2+\dots+x_n)$ i njegovu geometrisku sredinu kao broj $G=\sqrt[n]{x_1x_2\dots x_n}$. Dokažite da vrijedi nejednakost

$$\sqrt[n]{x_1x_2\dots x_n}\le\frac{1}{n}(x_1+x_2+\dots+x_n).$$

[(Nejednakost prelazi u jednakost akko je $x_1=x_2=\dots=x_n$]

Rj. $A=\frac{1}{k}(x_1+x_2+\dots+x_k)$, $G=\sqrt[k]{x_1x_2\dots x_k}$, $G\le A$, $k\in N$

BAZA INDUKCIJE

$k=1$: $\sqrt[1]{x_1}\le\frac{1}{1}(x_1)$ tj. $x_1\le x_1$ Za $k=1$ nejednakost je tačna.

INDUKCISKI KORAK

Pretpostavimo da je $\sqrt[k]{x_1x_2\dots x_k}\le\frac{1}{k}(x_1+x_2+\dots+x_k)$ za $k=1,2,\dots,n$.

Dokažimo da je $\sqrt[n+1]{x_1x_2\dots x_{n+1}}\le\frac{1}{n+1}(x_1+x_2+\dots+x_{n+1})$.

Ne gubeći općost dokaza možemo smatrati da je $x_1\le x_2\le\dots\le x_{n+1}$ ...(*)

Označimo sa $A=\frac{1}{n+1}(x_1+x_2+\dots+x_{n+1})$, i sa $G=\sqrt[n+1]{x_1\cdot x_2\cdot\ \dots\ \cdot x_{n+1}}$.

Primjetimo da vrijedi

$$x_1=\frac{1}{n+1}\underbrace{(x_1+x_1+\dots+x_1)}_{(n+1)\text{ puta}}\overset{(*)}{\le}A\overset{(*)}{\le}\frac{1}{n+1}(x_{n+1}+x_{n+1}+\dots+x_{n+1})=x_{n+1}\quad\dots(**)$$

Posmatrajmo sada sljedeće brojeve $x_2, x_3, \dots, x_n, x_1+x_{n+1}-A$.

$(**) \Rightarrow \quad A - x_1 \geq 0 \quad i \quad x_{n+1} - A \geq 0 \quad i \quad x_1 + x_{n+1} - A \geq 0$

Pa je

$$(A-x_1)\cdot(x_{n+1}-A) \geq 0$$

$$\underline{Ax_{n+1}}\ \underline{-A^2} - x_1x_{n+1} + \underline{Ax_1} \geq 0$$

$$A(x_1+x_{n+1}-A) \geqslant x_1x_{n+1} \quad .$$

Na n brojeva $x_2, x_3, \dots, x_n, \underline{x_1+x_{n+1}-A}$ primjenimo indukcisku pretpostavku, dobidemo:

$$\frac{1}{n}(x_2+x_3+\dots+x_n+\overbrace{x_1+x_{n+1}-A}) \geqslant \sqrt[n]{x_2\cdot x_3\cdot\dots\cdot x_n\cdot(x_1+x_{n+1}-A)}$$

$$\left[\frac{1}{n}(x_1+x_2+\dots+x_n+x_{n+1}-A)\right]^n \geqslant x_2\cdot x_3\cdot\dots\cdot x_n\cdot(x_1+x_{n+1}-A)$$

$$\left[\frac{1}{n}(x_1+x_2+\dots+x_n+x_{n+1}-A)\right]^n = \left[\frac{1}{n}\left(x_1+x_2+\dots+x_{n+1}-\frac{1}{n+1}(x_1+x_2+\dots+x_{n+1})\right)\right]^n =$$

$$x_1 - \frac{x_1}{n+1} = \frac{x_1(n+1)-x_1}{n+1} = \frac{x_1\cdot n}{n+1}$$

$$x_2 - \frac{x_2}{n+1} = \frac{x_2n+x_2-x_2}{n+1} = \frac{x_2\cdot n}{n+1}$$

$$\vdots$$

$$= \left[\frac{1}{n}\left(\frac{n}{n+1}(x_1+x_2+\dots+x_{n+1})\right)\right]^n =$$

$$= \left[\frac{1}{n+1}(x_1+x_2+\dots+x_{n+1})\right]^n = A^n$$

Pa imamo

$$A^n \geqslant x_2\cdot x_3\cdot\dots\cdot x_n(x_1+x_{n+1}-A) \quad /\cdot A$$

$$A^{n+1} \geqslant x_2\cdot x_3\cdot\dots\cdot x_n\cdot A(x_1+x_{n+1}-A) \overset{\text{kako je } A(x_1+x_{n+1}-A)\geqslant x_1x_{n+1}}{\geqslant} x_1\cdot x_2\cdot\dots\cdot x_{n+1} \Rightarrow$$

$$\Rightarrow A \geqslant \sqrt[n+1]{x_1\cdot x_2\cdot\dots\cdot x_{n+1}} \Rightarrow \frac{1}{n+1}(x_1+x_2+\dots+x_{n+1}) \geqslant \sqrt[n+1]{x_1\cdot x_2\cdot\dots\cdot x_{n+1}}$$

što je i trebalo dobiti

ZAKLJUČAK

Nejednakost je tačna za sve prirodne brojeve n.

⑭ Metodom matematičke indukcije dokazati:

a) $1+2+\dots+n = \frac{1}{2}n(n+1)$, n je prirodan broj.

b) $1^2+2^2+\dots+n^2 = \frac{1}{6}n(n+1)(2n+1)$, $n\in\mathbb{N}$.

c) $1+3+\dots+(2n-1) = n^2$, $n\in\mathbb{N}$.

d) $2+4+6+\dots+(2n) = n(n+1)$, $n\in\mathbb{N}$.

e) $1+a+a^2+\dots+a^n = \frac{a^{n+1}-1}{a-1}$, $a\neq 1$, $n\in\mathbb{N}$,

(#) Dokazati matematičkom indukcijom tvrdnju
$5|(n^5-n)$, $n\in\mathbb{N}$.

Rj. $5|(k^5-k)$, $k\in\mathbb{N}$ (ovo čitamo: pet djeli k^5-k gdje je k neki prirodan broj) (ili k^5-k je djeljivo sa 5)

BAZA INDUKCIJE

$k=1$: $5|(1^5-1)$ tj. $5|0$ 5 djeli 0 tj. $0=5\cdot s$ gdje je s neki broj iz $\mathbb{N}$.

Tvrdnja je tačna za $k=1$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

KORAK INDUKCIJE

Pretpostavimo da je tvrdnja $5|(k^5-k)$ tačna za sve brojeve od 1 do n. Na osnovu ove pretpostavke dokažimo da $5|(n+1)^5-(n+1)$

$$\begin{aligned}(n+1)^5-(n+1) &= n^5+5n^4+10n^3+10n^2+5n\underline{+1}-n\underline{-1}=\\ &= n^5+5n^4+10n^3+10n^2+5n-n=\\ &= \underbrace{(n^5-n)}_{\text{ovo je prema pretpostavci djeljivo sa 5}}+\underbrace{5(n^4+2n^3+2n^2+n)}_{\text{ovo je djeljivo sa 5 (vidi se)}}\end{aligned}$$

Prema tome $5|(n+1)^5-(n+1)$ što je i trebalo pokazati

ZAKLJUČAK

Tvrdnja je tačna za sve prirodne brojeve.

(#) Dokazati matematičkom indukcijom da važi

$$1-x+x^2-x^3+\ldots+(-1)^{n-1}x^{n-1}=\frac{1+(-1)^{n-1}x^n}{1+x}\quad (x\in\mathbb{R},\ n\in\mathbb{N}).$$

Rj. BAZA INDUKCIJE

Dokažimo da je jednakost tačna za broj 1

$$1=\frac{1+(-1)^0x^1}{1+x}=\frac{1+x}{1+x}=1$$

Jednakost je tačna za broj 1.

KORAK INDUKCIJE

Pretpostavimo da je jednakost $1-x+x^2-\ldots+(-1)^{k-1}x^{k-1}=\frac{1+(-1)^{k-1}x^k}{1+x}$ tačna za sve brojeve k od 1 do n i na osnovu ove pretpostavke dokažimo da je jednakost tačna za n+1 tj. dokažimo

$$1-x+x^2-x^3+\ldots+(-1)^{n-1}x^{n-1}+(-1)^nx^n=\frac{1+(-1)^nx^{n+1}}{1+x}$$

$$1-x+x^2-x^3+\ldots+(-1)^{n-1}x^{n-1}+(-1)^nx^n\overset{\text{na osnovu pretpostavke}}{=}\frac{1+(-1)^{n-1}x^n}{1+x}+(-1)^nx^n=$$

$$=\frac{1+(-1)^{n-1}x^n+(-1)^nx^n\cdot(1+x)}{1+x}=\frac{1+[(-1)^{n-1}+(-1)^n(1+x)]x^n}{1+x}=$$

$$=\frac{1+[(-1)^{n-1}(1+(-1)(1+x)]x^n}{1+x}=\frac{1+[(-1)^{n-1}\cdot(1-1-x]x^n}{1+x}=$$

$$=\frac{1+(-1)^{n-1}\cdot(-1)x\cdot x^n}{1+x}=\frac{1+(-1)^nx^{n+1}}{1+x}$$

što je i trebalo dobiti

Jednakost je tačna za n+1

ZAKLJUČAK

Jednakost je tačna za sve prirodne brojeve.

(#) Matematičkom indukcijom dokazati da je $3\cdot 5^{2n+1} + 2^{3n+1}$ djeljivo sa 17 za svaki prirodan broj n.

Rj. $3\cdot 5^{2k+1} + 2^{3k+1}$ djeljivo sa 17, $k \in \mathbb{N}$

BAZA INDUKCIJE

$k=1$: $3\cdot 5^{2+1} + 2^{3+1} = 3\cdot 5^{3} + 2^{4} = 3\cdot 125 + 16 = 375 + 16 = 391$

$391 : 17 = 23$
$\underline{34}$
51
$\underline{51}$
$==$

Broj 391 jest djeljiv sa 17
Tvrdnja je tačna za broj 1

KORAK INDUKCIJE

Pretpostavimo da je $3\cdot 5^{2k+1} + 2^{3k+1}$ djeljivo sa 17 za svaki broj k od 1 do n. Uz pomoć ove pretpostavke dokažimo da je $3\cdot 5^{2(n+1)+1} + 2^{3(n+1)+1}$ djeljivo sa 17.

$$3\cdot 5^{2(n+1)+1} + 2^{3(n+1)+1} = 3\cdot 5^{2n+3} + 2^{3n+4} = 3\cdot 5^{2n+1}\cdot 5^{2} + 2^{3n+1}\cdot 2^{3} =$$

$$= 25\left(3\cdot 5^{2n+1}\right) + 8\left(2^{3n+1}\right) = 17\left(3\cdot 5^{2n+1}\right) + 8\left(3\cdot 5^{2n+1}\right) +$$

$$+ 8\left(2^{3n+1}\right) = \underbrace{17\cdot\left(3\cdot 5^{2n+1}\right)}_{\text{vidimo da je ovo djeljivo sa 17}} + 8\underbrace{\left(3\cdot 5^{2n+1} + 2^{3n+1}\right)}_{\text{na osnovu pretpostavke ovo je djeljivo sa 17}}$$

Prema tome tvrdnja je tačna za n+1, tj. $3\cdot 5^{2(n+1)+1} + 2^{3(n+1)+1}$ je djeljivo sa 17.

ZAKLJUČAK

Tvrdnja je tačna za svaki prirodan broj n.

Ⓧ Dokazati metodom matematičke indukcije da za sve prirodne brojeve n važi

$$\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{n^2+3n+2}=\frac{n}{2n+4}.$$

Rj. $\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{k^2+3k+2}=\frac{k}{2k+4}$, k je pozitivan cijeli br.

BAZA INDUKCIJE

$k=1$: $\frac{1}{6}=\frac{1}{2\cdot 1+4} \Rightarrow \frac{1}{6}=\frac{1}{6}$ jednakost je tačna za $k=1$.

INDUKCISKI KORAK

Pretpostavimo da je jednakost tačna za $k=1,2,\ldots,n$, tj. $\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{k^2+3k+2}=\frac{k}{2k+4}$, $k=1,2,\ldots,n$.

Na osnovu ove pretpostavke dokažimo da je jednakost tačna za $n+1$ tj. da je

$$\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{(n+1)^2+3(n+1)+2}=\frac{n+1}{2(n+1)+4}$$

$(n+1)^2=n^2+2n+1$

$3(n+1)=3n+3$

ili drugačije napisano $\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{n^2+5n+6}=\frac{n+1}{2n+6}$

$$\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{n^2+3n+2}+\frac{1}{n^2+5n+5}\overset{\text{na osnovu pretpostavke}}{=}\frac{n}{2n+4}+\frac{1}{n^2+5n+6}$$

$n^2+5n+6=0$

$D=25-24=1$

$n_{1,2}=\frac{-5\pm 1}{2}$

$n_1=\frac{-6}{2}=-3 \quad n_2=\frac{-4}{2}=-2$

$$=\frac{n}{2(n+2)}{}^{\cdot(n+3)}_{\cdot(n+3)}+\frac{1}{(n+2)(n+3)}{}^{\cdot 2}_{\cdot 2}=\frac{n(n+3)+2}{2(n+2)\cdot(n+3)}$$

$$=\frac{n^2+3n+2}{2(n+2)(n+3)}=\frac{\cancel{(n+2)}(n+1)}{2\cancel{(n+2)}(n+3)}=\frac{n+1}{2n+6}$$ što je i trebalo dobiti

ZAKLJUČAK

Jednakost je tačna za sve prirodne brojeve.

(#) Dokazati metodom matematičke indukcije da za sve prirodne brojeve n važi

$$\frac{1^2}{1\cdot3}+\frac{2^2}{3\cdot5}+\dots+\frac{n^2}{(2n-1)(2n+1)}=\frac{n(n+1)}{2(2n+1)}$$

tj. $$\frac{1^2}{1\cdot3}+\frac{2^2}{3\cdot5}+\dots+\frac{k^2}{(2k-1)(2k+1)}=\frac{k(k+1)}{2(2k+1)}.$$

BAZA INDUKCIJE

$k=1$: $$\frac{1^2}{1\cdot3}=\frac{1\cdot(1+1)}{2(2+1)} \quad \text{tj.} \quad \frac{1}{3}=\frac{2}{2\cdot3}=\frac{1}{3}$$

Jednakost je tačna za broj 1

KORAK INDUKCIJE

Pretpostavimo da je jednakost $\frac{1^2}{1\cdot3}+\frac{2^2}{3\cdot5}+\dots+\frac{k^2}{(2k-1)(2k+1)}=\frac{k(k+1)}{2(2k+1)}$ tačna za svako k od 1 do n.
Na osnovu ove pretpostavke dokažimo da je jednakost tačna za $n+1$ tj. dokažimo da je

$$\frac{1^2}{1\cdot3}+\frac{2^2}{3\cdot5}+\dots+\frac{n^2}{(2n-1)(2n+1)}+\frac{(n+1)^2}{(2n+1)(2n+3)}=\frac{(n+1)(n+2)}{2(2n+3)}.$$

$$\frac{1^2}{1\cdot3}+\frac{2^2}{3\cdot5}+\dots+\frac{n^2}{(2n-1)(2n+1)}+\frac{(n+1)^2}{(2n+1)(2n+3)}\overset{\text{na osnovu pretpostavke}}{=}$$

$$=\frac{n(n+1)}{2(2n+1)}+\frac{(n+1)^2}{(2n+1)(2n+3)}=\frac{n(n+1)(2n+3)+(n+1)^2\cdot2}{2(2n+1)(2n+3)}=$$

$$=\frac{(n+1)\left[n(2n+3)+2(n+1)\right]}{2(2n+1)(2n+3)}=\frac{(n+1)(2n^2+3n+2n+2)}{2(2n+1)(2n+3)}=\frac{(n+1)(2n^2+5n+2)}{2(2n+1)(2n+3)}$$

$$=\frac{(n+1)(2n+1)(n+2)}{2(2n+1)(2n+3)}=\frac{(n+1)(n+2)}{2(2n+3)}$$ što je i trebalo dobiti

Jednakost je tačna za $n+1$.

ZAKLJUČAK:

Jednakost je tačna za sve prirodne brojeve.

#) Dokazati metodom matematičke indukcije da vrijedi za sve $n\in\{2,3,4,\ldots\}$:

$$\frac{1}{\log_x 2\cdot\log_x 4}+\frac{1}{\log_x 4\cdot\log_x 8}+\ldots+\frac{1}{\log_x 2^{n-1}\cdot\log_x 2^{n}}=\left(1-\frac{1}{n}\right)\frac{1}{(\log_x 2)^2}.$$

Rj. postavka zadatka.

$$\frac{1}{\log_x 2\cdot\log_x 4}+\frac{1}{\log_x 4\cdot\log_x 8}+\ldots+\frac{1}{\log_x 2^{k-1}\cdot\log_x 2^{k}}=\left(1-\frac{1}{k}\right)\frac{1}{(\log_x 2)^2},\quad k=2,3,\ldots$$

BAZA INDUKCIJE

$$k=2:\quad \frac{1}{\log_x 2\cdot\log_x 4}=\left(1-\frac{1}{2}\right)\cdot\frac{1}{\log_x 2\cdot\log_x 2}=\frac{1}{2}\cdot\frac{1}{\log_x 2\cdot\log_x 2}=\frac{1}{\log_x 2\cdot 2\log_x 2}=\frac{1}{\log_x 2\cdot\log_x 4}$$

Tvrdnja je tačna za $k=2$.

KORAK INDUKCIJE

Pretpostavimo da je jednakost $\frac{1}{\log_x 2\cdot\log_x 4}+\frac{1}{\log_x 4\cdot\log_x 8}+\ldots+\frac{1}{\log_x 2^{k-1}\log_x 2^{k}}=\left(1-\frac{1}{k}\right)\frac{1}{(\log_x 2)^2}$ tačna za svako $k=2,3,\ldots,n$.

Na osnovu ove pretpostavke dokažimo da je

$$\frac{1}{\log_x 2\cdot\log_x 4}+\frac{1}{\log_x 4\cdot\log_x 8}+\ldots+\frac{1}{\log_x 2^{n-1}\cdot\log_x 2^{n}}+\frac{1}{\log_x 2^{n}\log_x 2^{n+1}}=\left(1-\frac{1}{n+1}\right)\cdot\frac{1}{(\log_x 2)^2}$$

$$\frac{1}{\log_x 2\cdot\log_x 4}+\frac{1}{\log_x 4\cdot\log_x 8}+\ldots+\frac{1}{\log_x 2^{n-1}\cdot\log_x 2^{n}}+\frac{1}{\log_x 2^{n}\cdot\log_x 2^{n+1}}$$

na osnovu pretpostavke

$$=\left(1-\frac{1}{n}\right)\frac{1}{(\log_x 2)^2}+\frac{1}{\log_x 2^{n}\cdot\log_x 2^{n+1}}=\left(1-\frac{1}{n}\right)\frac{1}{(\log_x 2)^2}+\frac{1}{n\cdot(n+1)\log_x 2\cdot\log_x 2}$$

$$=\left(1-\frac{1}{n}\right)\frac{1}{(\log_x 2)^2}+\frac{1}{n(n+1)(\log_x 2)^2}=\left(1-\frac{1}{n}+\frac{1}{n(n+1)}\right)\frac{1}{(\log_x 2)^2}$$

$$=\left(1+\frac{-(n+1)+1}{n(n+1)}\right)\frac{1}{(\log_x 2)^2}=\left(1+\frac{-n}{n(n+1)}\right)\cdot\frac{1}{(\log_x 2)^2}=\left(1-\frac{1}{n+1}\right)\cdot\frac{1}{(\log_x 2)^2}$$

što je i trebalo dobiti

ZAKLJUČAK

Jednakost je tačna za sve brojeve $n\in\{2,3,4,\ldots\}$

(#) Dokazati matematičkom indukcijom tvrdnju

$7 \mid (n^7 - n)$, $n \in \mathbb{N}$.

R:

BAZA INDUKCIJE

Dokažimo da je tvrdnja tačna za broj 1.

$n=1$: $n^7 - n = 1^7 - 1 = 0$, $7 \mid 0$ (7 dijeli 0)

$0 = 7 \cdot 0$ Tvrdnja je tačna za broj 1.

~~KORAK~~ KORAK INDUKCIJE

Pretpostavimo da je tvrdnja tačna za brojeve od 1 do n tj. $7 \mid (k^7 - k)$ za $k = 1, 2, 3, \dots, n-1, n$. Na osnovu ove pretpostavke dokažimo da je tvrdnja tačna za $n+1$ tj: da $7 \mid [(n+1)^7 - (n+1)]$.

$$n^7 - n = n(n^6-1) = n(n^3-1)(n^3+1) = \underline{\underline{n}}(n-1)\underline{(n^2+n+1)}\,\underline{\underline{(n+1)}}(n^2-n+1)$$

$$(n+1)^7 - (n+1) = (n+1)\left[(n+1)^6 - 1\right] = (n+1)\left[(n+1)^3-1\right]\left[(n+1)^3+1\right] =$$

$$= (n+1)\left[(n+1)-1\right]\left[(n+1)^2+n+1+1\right]\left[(n+1)+1\right]\left[(n+1)^2-(n+1)+1\right]$$

$$= \underline{\underline{(n+1)}}\,\underline{\underline{n}}\,(n^2+3n+3)(n+2)\underline{(n^2+n+1)}$$

Pronađimo vezu između $(n-1)(n^2-n+1)$ i $(n^2+3n+3)(n+2)$

$$\left.\begin{aligned} &(n-1)(n^2-n+1) = n^3 - n^2 + n - n^2 + n - 1 = n^3 - 2n^2 + 2n - 1 \\ &(n+2)(n^2+3n+3) = n^3 + \underline{3n^2} + \underline{\underline{3n}} + \underline{2n^2} + \underline{\underline{6n}} + 6 = n^3 + 5n^2 + 9n + 6 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow (n+2)(n^2+3n+3) = (n-1)(n^2-n+1) - 7n^2 - 7n - 7$$

pa imamo: $(n+1)^7 - (n+1) = (n+1)n(n^2+n+1)\left[(n-1)(n^2-n+1) - 7(n^2+n+1)\right]$

$$= (n+1)n(n^2+n+1)(n-1)(n^2-n+1) - 7(n+1)n(n^2+n+1)^2$$

$$= \underbrace{(n^7 - n)}_{A} - \underbrace{7n(n+1)(n^2+n+1)^2}_{B}$$

$$\left.\begin{aligned} &\text{A je prema pretpostavci djeljivo sa 7} \\ &\text{B je očigledno djeljivo sa 7} \end{aligned}\right\} \Rightarrow (n+1)^7 - (n+1) \text{ je djeljivo sa 7 tj. } 7 \mid (n+1)^7 - (n+1)$$

ZAKLJUČAK

Tvrdnja $7 \mid (n^7 - n)$ je tačna za sve prirodne brojeve

Binomna formula (obrazac)

$n!$ - čitamo n faktorijel

n je prirodan broj (pozitivan cijeli broj) $(1, 2, 3, \dots)$

$$n! = 1\cdot 2\cdot 3\cdot \ldots \cdot (n-1)\cdot n$$

$\binom{n}{k}$ - čitamo n nad k

$$\binom{n}{k} = \frac{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot[n-(k-2)][n-(k-1)]}{1\cdot 2\cdot\ldots\cdot(k-1)\cdot k}, \quad \binom{n}{k} = \frac{n!}{k!\,(n-k)!}, \quad n\geq k$$

ako je $k>n$ $\binom{n}{k}=0$, $\binom{n}{0}=\binom{n}{n}=1$, $\binom{n}{k}=\binom{n}{n-k}$

npr. $\binom{7}{3} = \frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3} = 35$, $\binom{3}{5} = 0$,

$\binom{21}{18} = \binom{21}{3} = \frac{\overset{7}{\cancel{21}}\cdot\overset{10}{\cancel{20}}\cdot 19}{1\cdot\underset{1}{\cancel{2}}\cdot\underset{1}{\cancel{3}}} = 1\,330$, $\binom{7}{7} = 1$.

Za svaka dva realna broja a i b, i za svaki prirodan broj n važi:

koeficijent drugog člana

koeficijent prvog člana

koeficijent posljednjeg člana

$$(a+b)^n = \underbrace{\binom{n}{0}a^n}_{} + \underbrace{\binom{n}{1}a^{n-1}b}_{} + \ldots + \binom{n}{n-1}ab^{n-1} + \underbrace{\binom{n}{n}b^n}_{}$$

vrijednost prvog sabirnika

vrijednost drugog sabirnika

vrijednost posljednjeg sabirnik

$$= \sum_{k=0}^{n} \boxed{\binom{n}{k}} a^{n-k} b^k$$

vrijednost sabirnika

binomni obrazac

koeficijenti binomnog razvoja

$$n! = (n-1)!\cdot n$$

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

$$0! = 1! = 1$$

$$(2n)!! = 2\cdot 4\cdot 6\cdot \ldots \cdot (2n)$$

$$\binom{n+1}{k+1} = \binom{n}{k}\cdot\frac{n+1}{k+1}$$

$$(2n-1)!! = 1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)$$

① Razviti izraz $(2x-3)^5$.

Rj. $$(2x-3)^5=\binom{5}{0}(2x)^5+\binom{5}{1}(2x)^4(-3)+\binom{5}{2}(2x)^3(-3)^2+\binom{5}{3}(2x)^2(-3)^3+$$
$$+\binom{5}{4}(2x)(-3)^4+\binom{5}{5}(-3)^5=2^5\cdot x^5+5\cdot(-3)\cdot 2^4\cdot x^4+10\cdot 2^3\cdot 9x^3$$

$$\boxed{\binom{5}{5}=\binom{5}{0}=1,\quad \binom{5}{1}=\binom{5}{4}=\frac{5}{1}=5,\quad \binom{5}{3}=\binom{5}{2}=\frac{5\cdot 4}{1\cdot 2}=10}$$

$$+10\cdot 4\cdot(-3)^3x^2+5\cdot 2\cdot 81x+1\cdot 81\cdot(-3)=$$
$$=32x^5-240x^4+720x^3-1080x^2+810x-243$$

② U razvoju binoma $\left(\sqrt{x}+\frac{1}{\sqrt[4]{x}}\right)^6$ odrediti član koji ne sadrži x.

Rj. $$\left(\sqrt{x}+\frac{1}{\sqrt[4]{x}}\right)^6=\sum_{k=0}^{6}\binom{6}{k}\left(\sqrt{x}\right)^{6-k}\Big(\underbrace{\frac{1}{\sqrt[4]{x}}}_{x^{-\frac{1}{4}}}\Big)^k=\sum_{k=0}\binom{6}{k}x^{\frac{6-k}{2}}\cdot x^{-\frac{k}{4}}$$
$$=\sum_{k=0}^{6}\binom{6}{k}x^{3-\frac{k}{2}-\frac{k}{4}}=\sum_{k=0}^{6}\binom{6}{k}x^{3-\frac{3k}{4}}$$

Tražimo član koji ne sadrži x, tj. član koji sadrži x^0.

$3-\frac{3k}{4}=0$ $\qquad$ $k=4$

$12-3k=0$ $\qquad$ Peti član u razvoju binoma ne sadrži x.

③ Odrediti koji član razvoja binoma $\left(\frac{3}{4}\sqrt[3]{a^2}+\frac{2}{3}\sqrt{a}\right)^{12}$ sadrži a^7.

Rj. $$\left(\frac{3}{4}\sqrt[3]{a^2}+\frac{2}{3}\sqrt{a}\right)^{12}=\sum_{k=0}^{12}\binom{12}{k}\left(\frac{3}{4}\sqrt[3]{a^2}\right)^{12-k}\cdot\left(\frac{2}{3}\sqrt{a}\right)^k=$$
$$=\sum_{k=0}^{12}\binom{12}{k}\left(\frac{3}{4}\right)^{12-k}a^{\frac{2(12-k)}{3}}\cdot\left(\frac{2}{3}\right)^k\cdot a^{\frac{k}{2}}=\sum_{k=0}^{12}\binom{12}{k}\left(\frac{3}{4}\right)^{12-k}\cdot\left(\frac{2}{3}\right)^k\cdot a^{8-\frac{2k}{3}+\frac{k}{2}}$$
$$=\sum_{k=0}^{12}\binom{12}{k}\left(\frac{3}{4}\right)^{12-k}\left(\frac{2}{3}\right)^k a^{8-\frac{k}{6}}$$

Tražimo član koji sadrži a^7.

$8-\frac{k}{6}=7 \quad |\cdot 6$ $\qquad k=6$

$48-k=42$ $\qquad$ Sedmi član u razvoju binoma sadrži a^7.

④ Izračunati $\binom{n}{0}+\binom{n}{1}+\dots+\binom{n}{n-1}+\binom{n}{n}$.

Rj.
$$2^n=(1+1)^n=\sum_{k=0}^{n}\binom{n}{k}1^{n-k}\cdot 1^k=\sum_{k=0}^{n}\binom{n}{k}=\binom{n}{0}+\binom{n}{1}+\dots+\binom{n}{n}$$

Prema tome $\binom{n}{0}+\binom{n}{1}+\dots+\binom{n}{n-1}+\binom{n}{n}=2^n$.

⑤ Koliko racionalnih članova ima u razvoju $(\sqrt{2}+\sqrt[4]{3})^{100}$.

Rj. Koji brojevi se zovu racionalni brojevi? = Razlomci, bez $\sqrt{\ }$

$$(\sqrt{2}+\sqrt[4]{3})^{100}=\sum_{k=0}^{100}\binom{100}{k}(\sqrt{2})^{100-k}\cdot(\sqrt[4]{3})^{k}=\sum_{k=0}^{100}\binom{100}{k}2^{50-\frac{k}{2}}\cdot 3^{\frac{k}{4}}$$

U našem slučaju da bi član bio racionalan treba da su $50-\frac{k}{2}$ i $\frac{k}{4}$ cijeli brojevi, k uzima vrijednosti od 0 do 100.

$50-\frac{k}{2}$ će biti cijeli broj ako je $\frac{k}{2}$ cijeli broj, $\Rightarrow$ broj k je iz skupa $A=\{0,2,4,6,\dots,90,92,94,96,98,100\}$

$\frac{k}{4}$ će biti cijeli broj ako je k iz skupa $B=\{0,4,8,\dots,40,44,\dots,80,84,\dots,96,100\}$

$k\in A\cap B \Rightarrow$ 26 racionalnih članova ima u razvoju $(\sqrt{2}+\sqrt[4]{3})^{100}$

⑥ Naći članove u razvoju $(\sqrt[4]{x^3}+\sqrt[3]{x})^{10}$ koji su racionalni.

Rj. Racionalni brojevi? (svi brojevi u obliku razlomka)(npr. $\frac{73}{5}$)

$$(\sqrt[4]{x^3}+\sqrt[3]{x})^{10}=\sum_{k=0}^{10}\binom{10}{k}(\sqrt[4]{x^3})^{10-k}\cdot(\sqrt[3]{x})^{k}=\sum_{k=0}^{10}\binom{10}{k}(x^{\frac{3}{4}})^{10-k}\cdot(x^{\frac{1}{3}})^{k}=$$

$$= \sum_{k=0}^{10} \binom{10}{k} x^{\frac{30-3k}{4}} \cdot x^{\frac{k}{3}} = \sum_{k=0}^{10} \binom{10}{k} x^{\frac{30-3k}{4}+\frac{k}{3}} = \sum_{k=0}^{10} \binom{10}{k} x^{\frac{90-5k}{12}}$$

U ovom slučaju, da bi član bio racionalan potrebno je i dovoljno da je $\frac{90-5k}{12}$ cijeli broj tj. da je $90-5k$ djeljivo sa 12. Brojevi djeljivi sa 12 su $\{0, 12, 24, 36, 48, 60, 72, 84, 96, \dots\}$.

$\overset{k=1:}{90-5=85}$, $\overset{k=2:}{90-10=80}$, $\overset{k=3:}{90-15=75}$, $\overset{k=4:}{90-20=70}$, $\overset{k=5:}{90-25=65}$, $\overset{k=6:}{90-30=\underline{60}}$,

$\overset{k=7:}{90-35=55}$, $\overset{k=8:}{90-40=50}$, $\overset{k=9:}{90-45=45}$, $\overset{k=10:}{90-50=40}$, $\overset{k=0:}{90-0=90}$

2 član

prvi član

sedmi član

Sedmi član u razvoju je racionalan.

⑦ Naći članove u razvoju $(\sqrt[5]{3}+\sqrt[7]{2})^{20}$ koji nisu racionalni.

Rj. Kakvi brojevi su iracionalni brojevi? [npr. $\sqrt{2}, \sqrt{3}, \dots, \sqrt{73}, \dots$]

$$(\sqrt[5]{3}+\sqrt[7]{2})^{20} = \sum_{k=0}^{20} \binom{20}{k} (\sqrt[5]{3})^{20-k} (\sqrt[7]{2})^{k} = \sum_{k=0}^{20} \binom{20}{k} 3^{\frac{20-k}{5}} \cdot 2^{\frac{k}{7}}$$

Naći ćemo prvo koji članovi su racionalni.

Potrebno je i dovoljno da su $\frac{20-k}{5}$ i $\frac{k}{7}$ cijeli brojevi za istu vrijednost broja k.

$\frac{k}{7}$ cijeli broj $\Rightarrow$ k je djeljiv sa 7

$\frac{20-k}{5} = 4 - \frac{k}{5}$ cijeli broj $\Rightarrow$ k je djeljiv sa 5

$k \in \{0, 1, 2, 3, \dots, 20\}$ jedini k koji je djeljiv sa 5 i sa 7 je $k=0$

Svi članovi osim prvog nisu racionalni.

⑧ Za koju vrijednost promjenjive x u binomnom razvoju $\left(3x - \frac{1}{9x^2}\right)^n$ četvrti sabirnik ima vrijednost (-1), ako je koeficijent uz predposljednji član razvoja jednak 8.

Rj. $\binom{n}{n-1} = 8$, $\binom{n}{n-1} = \binom{n}{1} = 8 \Rightarrow n = 8$

$$\left(3x - \frac{1}{9x^2}\right)^8 = \sum_{k=0}^{8} \binom{8}{k} (3x)^{8-k} \cdot \left(-\frac{1}{9x^2}\right)^k = \sum_{k=0}^{8} \binom{8}{k} 3^{8-k} \cdot x^{8-k} \cdot \left(-\frac{1}{9}\right)^k \cdot x^{-2k}$$

$$= \sum_{k=0}^{8} \binom{8}{k} 3^{8-k} \cdot (-1)^k \cdot 3^{-2k} \cdot x^{-2k+8-k} = \sum_{k=0}^{8} \binom{8}{k} 3^{8-3k} \cdot (-1)^k \cdot x^{8-3k}$$

vrijednost četvrtog sabirnika je -1, $\binom{8}{3} = \frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3} = 56$

$$\binom{8}{3} 3^{8-9}\cdot(-1)^3\cdot x^{8-9} = 56\cdot\frac{1}{3}\cdot(-1)\cdot\frac{1}{x} = -\frac{56}{3x}$$

$$-\frac{56}{3x} = -1 \Rightarrow x = \frac{56}{3}$$

Za $x=\frac{56}{3}$ četvrti sabirnik u binomnom razvoju ima vrijednost (-1).

⑨ **Odrediti koji** član razvoja binoma $\left(4\sqrt[5]{x}+\frac{\sqrt[3]{x}}{2}\right)^n$ sadrži $x^2\cdot\sqrt[5]{x^4}$ ako je zbir prva tri binomna koeficijenta jednak 56.

Rj. $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}=56$

$1+n+\frac{n(n-1)}{2}=56 \quad /\cdot 2$

$2+2n+n^2-n=112$

$n^2+n-110=0$

$D=1+440=441$

$n_{1,2}=\frac{-1\pm 21}{2}$

$n_1=-11 \quad n_2=10$

ovo rješenje otpada

$\frac{k}{3}-\frac{k}{5}=\frac{5k-3k}{15}$

$$\left(4\sqrt[5]{x}+\frac{1}{2}\sqrt[3]{x}\right)^{10} = \sum_{k=0}^{10}\binom{10}{k}\left(4\sqrt[5]{x}\right)^{10-k}\cdot\left(\frac{1}{2}\right)^k\cdot\left(\sqrt[3]{x}\right)^k = \sum_{k=0}^{10}\binom{10}{k}2^{20-2k}\cdot 2^{-k}\cdot x^{2-\frac{k}{5}}\cdot x^{\frac{k}{3}}$$

$$= \sum_{k=0}^{10}\binom{10}{k}2^{20-3k}\cdot x^{2+\frac{2k}{15}}, \quad x^2\sqrt[5]{x^4}=x^{2+\frac{4}{5}}, \quad \frac{2k}{15}=\frac{4}{5} \Rightarrow k=6$$

Sedmi član razvoja binoma sadrži $x^2\sqrt[5]{x^4}$.

⑩ Izračunati: $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\ldots+(-1)^n\binom{n}{n}$

⑪ Naći racionalne članove u razvoju $\left(\sqrt[5]{3}+\sqrt[7]{2}\right)^{24}$.

⑫ Odrediti član u razvijenom obliku binoma $\left(\sqrt[4]{a^2x}+\sqrt[5]{\frac{1}{ax^2}}\right)^{13}$ koji ne sadrži x.

⑬ Odrediti član koji sadrži $x^{8,5}$ u razvoju binoma $\left(\frac{1}{x\sqrt{x}}+\sqrt[3]{x^2}\right)^{16}$.

Rješenja:

10. 0 11. $k=14$ 12. $k=5$ 13. $k=15$

(14.) Naći vrijednost promjenjive x u razvoju $(x+x^{\log x})^5$ čiji je treći član razvoja binoma milion (1 000 000).

Rj.
$$(x+x^{\log x})^5 = \sum_{k=0}^{5}\binom{5}{k}x^{5-k}\cdot(x^{\log x})^k = \sum_{k=0}^{5}\binom{5}{k}\underbrace{x^{5-k}\cdot x^{k\log x}}_{x^{5-k+k\log x}}$$

Treći član razvoja ($k=2$) iznosi milion.

$$\binom{5}{2}x^{5-2+2\log x} = 1000\,000$$

$$\frac{5\cdot 4}{2}x^{3+2\log x} = 1\,000\,000 \quad /:10$$

$$x^{3+2\log x} = 100\,000 \quad /\log$$

$$\log x^{3+2\log x} = \log 100\,000$$

$$(3+2\log x)\cdot\log x = 5$$

$$2\log^2 x + 3\log x - 5 = 0$$

$$\log x = t$$

$$2t^2+3t-5=0$$

$$D = 9+40 = 49 \qquad t_{1,2} = \frac{-3\pm 7}{4}$$

$$t_1 = -\frac{10}{4} = -\frac{5}{2} \qquad t_2 = 1$$

$$\log x = -\frac{5}{2}$$

$$x = 10^{-\frac{5}{2}} = \frac{1}{10^{\frac{5}{2}}} = \frac{1}{\sqrt{10^2\cdot 10^2\cdot 10}} = \frac{1}{100\sqrt{10}}$$

$$\log x = 1$$

$$x = 10$$

Za vrijednosti $x=10$ ili $x=10^{-\frac{5}{2}}$ treći član razvoja ima vrijednost milion.

(15.) Zaokružite broj $(1{,}01)^7$ na tri decimalna mjesta.

Rj.
$$(1{,}01)^7 = (1+0{,}01)^7 = (1+10^{-2})^7 = \sum_{k=0}^{7}\binom{7}{k}1^{7-k}\cdot(10^{-2})^k$$

$$= \sum_{k=0}^{7}\binom{7}{k}10^{-2k} = \binom{7}{0}10^0 + \binom{7}{1}10^{-2} + \binom{7}{2}10^{-4} + \dots$$

$10^{-2} = 0{,}01$

$10^{-4} = 0{,}0001$

$10^{-6} = 0{,}000001$

$$\approx 1\cdot 1 + 7\cdot 0{,}01 + \frac{7\cdot 6}{1\cdot 2}\cdot 0{,}0001$$

$$= 1 + 0{,}07 + 0{,}0021 = 1{,}072$$

broj zaokružen na tri decimalna mjesta

⑯ Za svaka dva realna broja a i b, i za svaki pozitivan cijeli broj n dokazati da važi:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k .$$

BINOMNA FORMULA

tj. $(a+b)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \ldots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k$

BAZA INDUKCIJE

$k=1$: $(a+b)^1 = \binom{1}{0}a^1 + \binom{1}{1}b^1$ tj. $a+b=a+b$

Za $k=1$ jednakost je tačna

INDUKCISKI KORAK

Pretpostavimo da je $(a+b)^k = \sum_{i=0}^{k}\binom{k}{i}a^{k-i}b^i$ za $k=1,2,\ldots,n$.

Na osnovu ove pretpostavke dokažimo da vrijedi

$$(a+b)^{n+1} = \sum_{i=0}^{n+1}\binom{n+1}{i}a^{(n+1)-i}b^{n+1} \quad \text{tj.}$$

$$(a+b)^{n+1} = \binom{n+1}{0}a^{n+1} + \binom{n}{1}a^n b + \ldots + \binom{n+1}{n}ab^n + \binom{n+1}{n+1}b^{n+1}$$

$$(a+b)^{n+1} = (a+b)\cdot(a+b)^n \overset{\text{na osnovu pretpostavke}}{=} (a+b)\left[\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n\right]$$

$$= \binom{n}{0}a^{n+1} + \binom{n}{1}a^n b + \ldots + \binom{n}{n-1}a^2b^{n-1} + \binom{n}{n}ab^n$$

$$+ \binom{n}{0}a^n b + \binom{n}{1}a^{n-1}b^2 + \ldots + \binom{n}{n-1}ab^n + \binom{n}{n}b^{n+1}$$

$$= \binom{n}{0}a^{n+1} + \left[\binom{n}{0} + \binom{n}{1}\right]a^n b + \left[\binom{n}{1} + \binom{n}{2}\right]a^{n-1}b^2 + \ldots +$$

$$+ \left[\binom{n}{n-1} + \binom{n}{n}\right]ab^n + \binom{n}{n}b^{n+1} =$$

$$\boxed{\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}}$$

$$= \binom{n+1}{0}a^{n+1} + \binom{n+1}{1}a^n b + \ldots + \binom{n+1}{n}ab^n + \binom{n+1}{n+1}b^{n+1}$$

što je i trebalo dobiti

ZAKLJUČAK

Jednakost $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$ je tačna za sve pozitivne cijele brojeve n.

⑰ Naći koeficijent uz x^7 u razvoju $(x^2-x+1)^5$.

Rj: $(x^2-x+1)^5=(1-x+x^2)^5=\sum_{k=0}^{5}\binom{5}{k}(1-x)^{5-k}\cdot(x^2)^k=$

$$=\sum_{k=0}^{5}\binom{5}{k}\left[\sum_{m=0}^{k}\binom{5-k}{m}1^{5-k-m}\cdot(-x)^m\right]x^{2k}=\sum_{k=0}^{5}\sum_{m=0}^{k}\binom{5}{k}\binom{5-k}{m}(-1)^m x^{2k+m}$$

Zanimaju nas koeficijenti uz x^7.

$k=0$, $m=\overline{0,5}$, x^{0+m}, za $k=0$ ne postoji x^7

$k=1$, $m=\overline{0,4}$, x^{2+m}, za $k=1$ ne postoji x^7.

$k=2$, $m=\overline{0,3}$, x^{4+m}, za $k=2$ i $m=3$ imamo x^7

$k=3$, $m=\overline{0,2}$, x^{6+m}, za $k=3$ i $m=1$ imamo x^7

$k=4$, $m=\overline{0,1}$, x^{8+m}, za $k=4$ i za $k=5$ ne postoji x^7

$$\binom{5}{2}\binom{3}{3}(-1)^3x^7+\binom{5}{3}\binom{2}{1}(-1)x^7=\frac{5\cdot4}{2}\cdot(-1)x^7+\frac{5\cdot4}{2}\cdot2\cdot(-1)x^7=-30x^7$$

Koeficijent uz x^7 iznosi -30.

⑱ Naći posljednje dvije cifre broja 13^9.

Rj: $13^9=(10+3)^9=\sum_{k=0}^{9}\binom{9}{k}10^{9-k}\cdot3^k=\sum_{k=0}^{7}\binom{9}{k}10^{9-k}3^k+\underbrace{\binom{9}{8}}_{9}10\,3^8+3^9$

$=\sum_{k=0}^{7}\binom{9}{k}10^{9-k}3^k+10\cdot3^{10}+3^9$,

$3^0=1$, $3^1=3$, $3^2=9$, $3^3=27$, $3^4=81$, $3^5=243$,

$3^{10}=5904\underline{9}$

$3^9=196\underline{\underline{83}}$

Posljednje dvije cifre broja 13^9 su 7 i 3.

⑲ Odrediti koeficijent uz x^8 u razvoju $\left(2x^3-\frac{3}{\sqrt{x}}\right)^5$.

⑳ Odrediti koeficijent uz x^4 u izrazu $(\sqrt{x}+x^2)^n$.

㉑ Ako je p prost broj a m cio broj dokazati, koristeći binomni obrazac da je m^p-m djeljivo sa p.

㉒ Koristeći binomni obrazac naći posljednje dvije cifre broja 7^9.

㉓ Naći maksimalan sabirak razvoja $\left(n+\frac{1}{n}\right)^{2n+1}$ gdje je n prirodan broj.

(#) Izračunati x ako je treći član u razvoju binoma $(x^{\log x}+x)^5$ jednak 100.

Rj. $$(x^{\log x}+x)^5=\sum_{k=0}^{5}\binom{5}{k}(x^{\log x})^{5-k}(x)^k$$

treći član je za $k=2$ tj. $\binom{5}{2}(x^{\log x})^3x^2=100$

$$\frac{5\cdot\cancel{4}^{2}}{1\cdot\cancel{2}}x^{3\log x}\cdot x^2=100 \quad /:10$$

$$x^{3\log x+2}=10 \quad /\log$$

$$\log x^{3\log x+2}=1$$

$$(3\log x+2)\log x=1$$

$$3\log^2x+2\log x-1=0$$

$\log x=t$

$3t^2+2t-1=0$

$D=4+12=16$

$t_{1,2}=\frac{-2\pm4}{6}$

$t_1=\frac{-2-4}{6}=-1 \quad t_2=\frac{2}{6}=\frac{1}{3}$

$\log x=-1$

$\log x=(-1)\log 10$

$\log x=\log 10^{-1}$

$x=\frac{1}{10}$ jedno rješenje

$\log x=\frac{1}{3}$

$\log x=\log 10^{\frac{1}{3}}$

$x=\sqrt[3]{10}$ drugo rješenje

(#) Odrediti član u razvoju binoma $\left(\sqrt[3]{\left(\frac{a}{b}\right)^2}+\frac{\sqrt[4]{b}}{\sqrt[8]{a^3}}\right)^{35}$ koji sadrži b^6.

Rj. $$\left(\sqrt[3]{\left(\frac{a}{b}\right)^2}+\frac{\sqrt[4]{b}}{\sqrt[8]{a^3}}\right)^{35}=\left(\frac{a^{\frac{2}{3}}}{b^{\frac{2}{3}}}+\frac{b^{\frac{1}{4}}}{a^{\frac{3}{8}}}\right)^{35}=\left(a^{\frac{2}{3}}b^{-\frac{2}{3}}+a^{-\frac{3}{8}}b^{\frac{1}{4}}\right)^{35}$$

$$=\sum_{k=0}^{35}\binom{35}{k}\left(a^{\frac{2}{3}}b^{-\frac{2}{3}}\right)^{35-k}\cdot\left(a^{-\frac{3}{8}}b^{\frac{1}{4}}\right)^k$$

Napisani izraz će sadržavati b^6 ako i samo ako je $\left(b^{-\frac{2}{3}}\right)^{35-k} \cdot b^{\frac{k}{4}} = b^6$ tj. $b^{\frac{-70+2k}{3}} \cdot b^{\frac{k}{4}} = b^6$

$$\Rightarrow b^{\frac{-70+2k}{3}+\frac{k}{4}} = b^6 \Rightarrow \frac{-70+2k}{3}+\frac{k}{4} = 6 \quad /\cdot 12$$

$$-280+8k+3k=72$$

$$11k=352$$

$k=32 \Rightarrow$ Tridesettreći član u razvoju binoma sadrži b^6.

(#) Naći sve racionalne članove u razvoju binoma $\left(\sqrt[6]{x}-\sqrt[9]{x}\right)^{42}$.

$$\left(\sqrt[6]{x}-\sqrt[9]{x}\right)^{42} = \sum_{k=0}^{42}\binom{42}{k}\left(\sqrt[6]{x}\right)^{42-k}\left(\sqrt[9]{x}\right)^{k} = \sum_{k=0}^{42}\binom{42}{k}x^{7-\frac{k}{6}}\cdot x^{\frac{k}{9}} =$$

$$= \sum_{k=0}^{42}\binom{42}{k}x^{7-\frac{k}{6}+\frac{k}{9}}$$

Da bi član u razvoju našeg binoma bio racionalan potrebno je i dovoljno da je $7-\frac{k}{6}+\frac{k}{9}$ cio broj. tj. da su $\frac{k}{6}$ i $\frac{k}{9}$ cijeli brojevi

$\frac{k}{6}$ je cio broj ako je $k\in\{0, 6, 12, 18, 24, 30, 36, 42\}$

$\frac{k}{9}$ je cio broj ako je $k\in\{0, 9, 18, 27, 36\}$

Racionalni članovi u razvoju binoma su za vrijednost $k=0$, $k=18$ i $k=36$.

Prvi, devetnaesti i tridesetsedmi član u razvoju binoma je racionalan.

(#) Naći sve racionalne članove u razvoju binoma $(\sqrt[6]{x}-\sqrt[9]{x})^{42}$.

j. $$(\sqrt[6]{x}-\sqrt[9]{x})^{42}=\sum_{k=0}^{42}\binom{42}{k}(\sqrt[6]{x})^{42-k}(\sqrt[9]{x})^{k}=\sum_{k=0}^{42}\binom{42}{k}x^{7-\frac{k}{6}}\cdot x^{\frac{k}{9}}=$$
$$=\sum_{k=0}^{42}\binom{42}{k}x^{7-\frac{k}{6}+\frac{k}{9}}$$

Da bi član u razvoju našeg binoma bio racionalan potrebno je i dovoljno da je $7-\frac{k}{6}+\frac{k}{9}$ cio broj. tj. da su $\frac{k}{6}$ i $\frac{k}{9}$ cijeli brojevi

$\frac{k}{6}$ je cio broj ako je $k\in\{0,6,12,18,24,30,36,42\}$

$\frac{k}{9}$ je cio broj ako je $k\in\{0,9,18,27,36\}$

Racionalni članovi u razvoju binoma su za vrijednost $k=0$, $k=18$ i $k=36$.

Prvi, devetnaesti i tridesetsedmi član u razvoju binoma je racionalan.

(#) Odrediti koji članovi u razvoju binoma $\left(\frac{\sqrt[4]{7}}{\sqrt[5]{2}} + \frac{1}{\sqrt[3]{5}}\right)^{23}$ su racionalni pa poslije toga naći njihovu vrijednost.

Rj.

$$\left(\frac{\sqrt[4]{7}}{\sqrt[5]{2}} + \frac{1}{\sqrt[3]{5}}\right)^{23} = \left(\frac{1}{\sqrt[3]{5}} + \frac{\sqrt[4]{7}}{\sqrt[5]{2}}\right)^{23} = \left(\frac{1}{5^{\frac{1}{3}}} + \frac{7^{\frac{1}{4}}}{2^{\frac{1}{5}}}\right)^{23} =$$

$$= \left(5^{-\frac{1}{3}} + 7^{\frac{1}{4}}\cdot 2^{-\frac{1}{5}}\right)^{23} = \sum_{k=0}^{23}\binom{23}{k}\left(5^{-\frac{1}{3}}\right)^{23-k}\cdot\left(7^{\frac{1}{4}}\cdot 2^{-\frac{1}{5}}\right)^{k} =$$

$$= \sum_{k=0}^{23}\binom{23}{k} 5^{\frac{-23+k}{3}}\cdot 7^{\frac{k}{4}}\cdot 2^{-\frac{k}{5}}$$

$7^{\frac{k}{4}}$ će biti racionalan za $k \in \{0, 4, 8, 12, 16, 20\}$

$2^{-\frac{k}{5}}$ će biti racionalan za $k \in \{0, 5, 10, 15, 20\}$

Prema tome $7^{\frac{k}{4}}\cdot 2^{-\frac{k}{5}}$ će biti racionalan za $k \in \{0, 20\}$

za $k=0$ imamo $5^{\frac{-23+0}{3}}$ da je iracionalan broj.

$k=20$ imamo $5^{\frac{-23+20}{3}} = 5^{-\frac{3}{3}} = 5^{-1} \in \mathbb{Q}$

Jedini racionalan član u razvoju binoma je dvadeset prvi član (za $k=20$).

Vrijednost ovog člana je $\binom{23}{20} 5^{-1}\cdot 7^{5}\cdot 2^{-4} = \frac{23\cdot 11\cdot 7\cdot 7^{5}}{5\cdot 2^{4}}$

$\binom{23}{20} = \binom{23}{3} = \frac{23\cdot \overset{11}{\cancel{22}}\cdot \overset{7}{\cancel{21}}}{1\cdot \cancel{2}\cdot \cancel{3}}$

$= \frac{23\cdot 11\cdot 7^{6}}{5\cdot 16}$ vrijednost dvadesetprvog člana

Da nisam obrno članove na početku $\left(\frac{\sqrt[4]{7}}{\sqrt[5]{2}} + \frac{1}{\sqrt[3]{5}}\right)^{23} = \left(\frac{1}{\sqrt[3]{5}} + \frac{\sqrt[4]{7}}{\sqrt[5]{2}}\right)^{23}$ dobili bi da je $k=3$ četvrti član

(#) Izračunati x ako se zna da treći član razvoja $\left(2\cdot\sqrt[x]{2^{-1}}+\frac{4}{\sqrt[4-x]{4}}\right)^6$ ima vrijednost 240.

$$\boxed{\sqrt[4-x]{4}=4^{\frac{1}{4-x}}}$$

Rj. $$\left(2\sqrt[x]{2^{-1}}+\frac{4}{\sqrt[4-x]{4}}\right)^6=\sum_{k=0}^{6}\binom{6}{k}\left(2\sqrt[x]{2^{-1}}\right)^{6-k}\left(\frac{4}{\sqrt[4-x]{4}}\right)^k=$$

$$=\sum_{k=0}^{6}\binom{6}{k}\left(2\cdot2^{-\frac{1}{x}}\right)^{6-k}\left(4\cdot4^{-\frac{1}{4-x}}\right)^k=\sum_{k=0}^{6}\binom{6}{k}\left(2^{1-\frac{1}{x}}\right)^{6-k}\left(4^{1-\frac{1}{4-x}}\right)^k$$

$k=0$ dobijemo prvi član
$k=1$ drugi član
$k=2$ treći član

$$\binom{6}{2}\left(2^{1-\frac{1}{x}}\right)^4\left(4^{1-\frac{1}{2}}\right)^2=240$$

$$\frac{6\cdot5}{2}\cdot\left(2^{\frac{x-1}{x}}\right)^4\cdot\left(4^{\frac{1}{2}}\right)^2=240$$

$$3\cdot5\cdot2^{\frac{4(x-1)}{x}}\cdot4=240\quad /:(4\cdot5)$$

$$3\cdot2^{\frac{4(x-1)}{x}}=12\quad /:3$$

$$2^{\frac{4(x-1)}{x}}=4$$

$$2^{\frac{4(x-1)}{x}}=2^2$$

$$\frac{4(x-1)}{x}=2\quad /\cdot x\ (x\neq0)$$

$$4x-4=2x$$

$$2x=4$$

$$x=2$$

Za $x=2$ treći član razvoja binoma ima vrijednost 240.

$$\boxed{(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k}$$

(#) Koliko ima racionalnih članova u razvoju binoma $(\sqrt[3]{4}+\sqrt[4]{3})^{120}$?

Rj. Koji su racionalni brojevi?

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

$$(\sqrt[3]{4}+\sqrt[4]{3})^{120}=\sum_{k=0}^{120}\binom{120}{k}(\sqrt[3]{4})^{120-k}(\sqrt[4]{3})^k=\sum_{k=0}^{120}\binom{120}{k}4^{\frac{120-k}{3}}\cdot 3^{\frac{k}{4}}=$$

$$=\sum_{k=0}^{120}\binom{120}{k}4^{40-\frac{k}{3}}\cdot 3^{\frac{k}{4}}$$

Da bi član bio racionalan, u posljednjem izrazu, potrebno je da je k djeljiv sa 3 (iz izraza $4^{40-\frac{k}{3}}$) i da je k djeljiv sa 4 (iz izraza $3^{\frac{k}{4}}$).

Kako je potrebno da je k djeljiv sa 3 i sa 4 to je potrebno da je k djeljiv i sa 12.

Brojevi djeljivi sa 12 iz intervala 0, 1, 2, ..., 120 su:

0, 12, 24, 36, 48, 60, 72, 84, 96, 108 i 120

Postoji 11 racionalnih članova u razvoju binoma.

Kompleksni brojevi

θ - THETA
φ - FI

$3+i,\ 2,\ 4i,\ 7-5i,\ i$

$z=a+bi$ je kompleksan broj, $a,b\in\mathbb{R}$

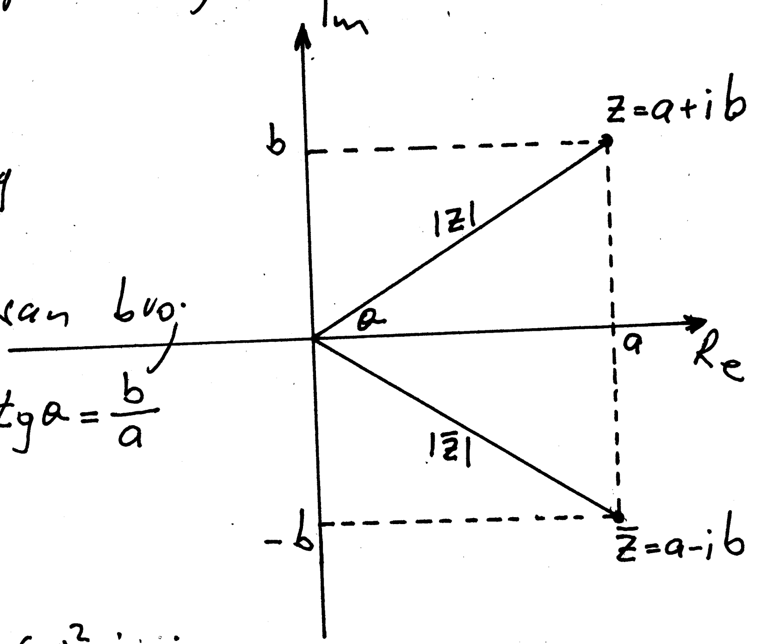
Možemo ga predstaviti u kompleksnoj ravni:

$z\in\mathbb{C}$

$|z|=\sqrt{a^2+b^2}$ modul kompleksnog broja

$\bar{z}=a-ib$ konjugovano kompleksan broj

$\cos\theta=\frac{a}{|z|},\quad \sin\theta=\frac{b}{|z|},\quad tg\theta=\frac{b}{a}$



$i^2 \overset{def.}{=} -1$

$i^4=i^2\cdot i^2=(-1)(-1)=1,\quad i^{33}=(i^2)^8\cdot i=(-1)^2\cdot i=i$

$i^8=(i^2)^4=(-1)^4=1,\quad i^{66}=(i^2)^{33}=(-1)^{33}=-1,\quad i^{67}=(i^2)^{33}\cdot i=(-1)^{33}\cdot i=-i$

$z=|z|(\cos\theta+i\sin\theta)$ trigonometrijski oblik kompleksnog broja

$z=|z|e^{i\theta},\ \theta\in[0,2\pi)$. Eulerov (eksponencijalni) oblik kompl. br.

$$\begin{matrix} z_1=|z_1|(\cos\varphi_1+i\sin\varphi_1) \\ z_2=|z_2|(\cos\varphi_2+i\sin\varphi_2) \end{matrix} \Rightarrow \begin{matrix} z_1=z_2 \text{ akko } |z_1|=|z_2| \text{ i } (\varphi_1=\varphi_2+2k\pi) \\ z_1\cdot z_2=|z_1||z_2|\left[\cos(\varphi_1+\varphi_2)+i\sin(\varphi_1+\varphi_2)\right] \\ \frac{z_1}{z_2}=\frac{|z_1|}{|z_2|}\left[\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2)\right],\ z_2\neq 0 \end{matrix}$$

$$z=|z|(\cos\theta+i\sin\theta)\ \Rightarrow\ z^n=|z|^n\left[\cos(n\theta)+i\sin(n\theta)\right].$$

Teorema: Jednačina $z^n=w$, gdje je w po volji odabran kompleksan broj različit od nule $(0\in\mathbb{C})$, ima tačno n različitih rješenja:

$$z_k=\sqrt[n]{|w|}\left[\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right]$$

gdje je φ $(z=|z|(\cos\varphi+i\sin\varphi))$ najmanji pozitivan ugao iz $[0,2\pi)$ a $k=0,1,\dots,n-1$.

① Zapisati u algebarskom obliku $(a+bi,\ a,b\in\mathbb{R})$ kompleksne brojeve a) $\frac{1}{1+i}$ b) $\frac{3+2i}{5-i}$.

Rj.

a) $\frac{1}{1+i}\cdot\frac{1-i}{1-i}=\frac{1-i}{1-i^2}=\frac{1-i}{2}=\frac{1}{2}-\frac{1}{2}i$ $\quad Re\left(\frac{1}{1+i}\right)=\frac{1}{2}$ $\quad Im\left(\frac{1}{1+i}\right)=-\frac{1}{2}$

imaginarni dio broj-a

realni dio broj-a

b) $\frac{3+2i}{5-i}=\frac{3+2i}{5-i}\cdot\frac{5+i}{5+i}=\frac{15+3i+10i+2i^2}{25-i^2}=\frac{13+13i}{26}=\frac{1}{2}+\frac{1}{2}i$. $\quad Re\left(\frac{3+2i}{5-i}\right)=\frac{1}{2}=Im\left(\frac{3+2i}{5-i}\right)$

② Odrediti kompleksan broj $z=a+bi$ koji zadovoljava jednačinu $|z|+z=2+i$.

Rj. $z=a+bi$

$|z|=\sqrt{a^2+b^2}$

$\sqrt{a^2+b^2}+a+bi=2+i \Rightarrow$

$$\sqrt{a^2+b^2}+a=2$$
$$bi=i$$
$$b=1$$

$\sqrt{a^2+1}+a=2$

$\sqrt{a^2+1}=2-a \quad /^2$

$a^2+1=(2-a)^2$

$a^2+1=4-4a+a^2$

$4a=3$

$a=\frac{3}{4}$

Traženi kompleksan broj je $z=\frac{3}{4}+i$.

③ Odrediti skup tačaka (x,y) ravni koje zadovoljavaju jednačinu $yi+(5i-x^2)i+5=0$.

Rj. $yi+5i^2-x^2i+5=0 \Rightarrow yi-x^2i=0 \Rightarrow (y-x^2)i=0$

$\Rightarrow y-x^2=0 \Rightarrow y=x^2$ Traženi skup tačaka je parabola s jednačinom $y=x^2$.

④ Napisati kvadratnu jednačinu kojoj su $z_1=1+3i$ i $z_2=1-3i$ korijeni (rješenja).

Rj. $(x-x_1)(x-x_2)=0$

$(x-(1-3i))(x-(1+3i))=0$

$(x-1+3i)(x-1-3i)=0$

$\boxed{x^2}\underline{\underline{-x}}\underline{-3ix}\ \underline{\underline{-x}}+1+\cancel{3i}\underline{+3ix}-\cancel{3i}-9i^2=0$

$x^2-2x+10=0$

Kvadratna jednačina kojoj su z_1 i z_2 korijeni je $x^2-2x+10=0$.

⑤ Brojeve $z_1 = -1+i$, $z_2 = \sqrt{3} - i$, $z_3 = -1-\sqrt{3}\,i$ predstaviti u trigonometriskom obliku, a zatim izračunati $\frac{z_1}{z_3}$, $z_1 \cdot z_2$ i $(z_2)^{2010}$.

Rj. $z = a+ib = |z|(\cos\theta + i\sin\theta)$, $|z| = \sqrt{a^2+b^2}$, $\cos\theta = \frac{a}{|z|}$, $\sin\theta = \frac{b}{|z|}$

Prisjetimo se vrijednosti sin, cos, tg i ctg

	$30° = \frac{\pi}{6}$ rad	$60° = \frac{\pi}{3}$ rad	$45° = \frac{\pi}{4}$ rad
$\sin\alpha$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$
$\cos\alpha$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$
$tg\,\alpha$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	1
$ctg\,\alpha$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	1

$$\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\cos\frac{\pi}{3} = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

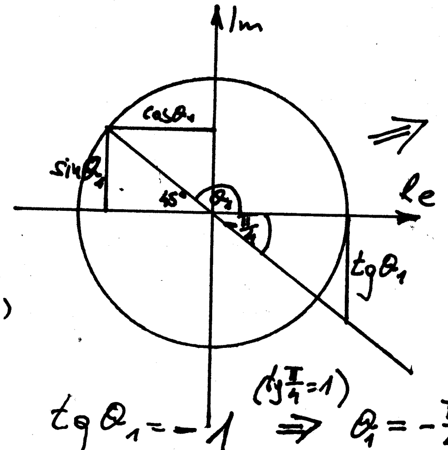
$z_1 = -1+i$

$|z_1| = \sqrt{1+1} = \sqrt{2}$

$\cos\theta_1 = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$,

$\sin\theta_1 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, $tg\,\theta_1 = -1 \overset{(tg\frac{\pi}{4}=1)}{\Rightarrow} \theta_1 = -\frac{\pi}{4}$

ili $\theta_1 = \frac{3\pi}{4}$



$\Rightarrow \theta_1 = \frac{3\pi}{4}$

$z_2 = \sqrt{3} - i$

$|z_2| = \sqrt{3+1} = 2$

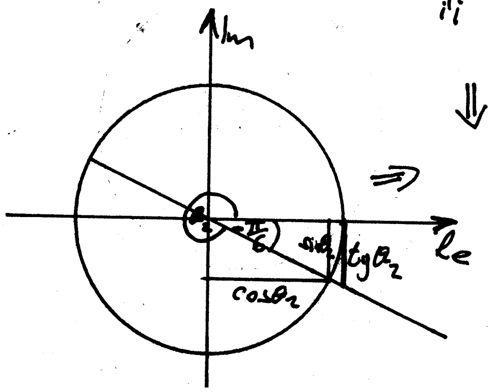
$\cos\theta_2 = \frac{\sqrt{3}}{2}$

$\sin\theta_2 = -\frac{1}{2}$

$tg\,\theta_2 = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} \overset{(tg\frac{\pi}{6}=\frac{\sqrt{3}}{3})}{\Rightarrow} \theta_2 = -\frac{\pi}{6}$

ili $\theta_2 = \frac{5\pi}{6}$

$\Downarrow$



$\Rightarrow \theta_2 = -\frac{\pi}{6}$

$z_3 = -1 - \sqrt{3}\,i$

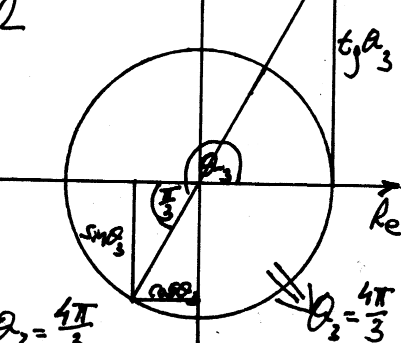
$|z_3| = \sqrt{1+3} = 2$

$\cos\theta_3 = -\frac{1}{2}$

$\sin\theta_3 = -\frac{\sqrt{3}}{2}$

$tg\,\theta_3 = \sqrt{3} \Rightarrow \theta_3 = \frac{\pi}{3}$

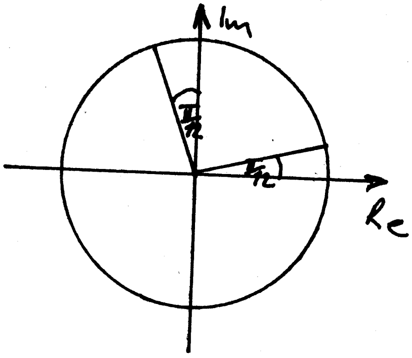
ili $\theta_3 = \frac{4\pi}{2}$



$$z_1 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$z_2 = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$z_3 = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$



$$\frac{z_1}{z_3} = \frac{\sqrt{2}}{2}\left(\cos\left(\frac{3\pi}{4} - \frac{4\pi}{3}\right) + i\sin\left(\frac{3\pi}{4} - \frac{4\pi}{3}\right)\right)$$

$$= \frac{\sqrt{2}}{2}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)$$

$$= \frac{\sqrt{2}}{2}\left(\cos\frac{7\pi}{12} - i\sin\frac{7\pi}{12}\right) =$$

$$= \frac{\sqrt{2}}{2}\left(-\sin\frac{\pi}{12} - i\cos\frac{\pi}{12}\right) = \frac{\sqrt{2}}{2}\left(-\frac{\sqrt{6}-\sqrt{2}}{4} - i\,\frac{\sqrt{6}+\sqrt{2}}{4}\right)$$

$$= \frac{-\sqrt{2}}{8}\left(\sqrt{6}-\sqrt{2} + i\left(\sqrt{6}+\sqrt{2}\right)\right)$$

$$z_1 \cdot z_2 = 2\sqrt{2}\left(\cos\left(\frac{3\pi}{4} + \left(-\frac{\pi}{6}\right)\right) + i\sin\left(\frac{3\pi}{4} + \left(-\frac{\pi}{6}\right)\right)\right) = 2\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right) =$$

$$= 2\sqrt{2}\left(-\sin\frac{\pi}{12} + i\cos\frac{\pi}{12}\right) = 2\sqrt{2}\left(-\frac{\sqrt{6}-\sqrt{2}}{4} + i\,\frac{\sqrt{6}+\sqrt{2}}{4}\right)$$

$$z_2^{2010} = 2^{2010}\left(\cos\left(2010\cdot\left(-\frac{\pi}{6}\right)\right) + i\sin\left(2010\cdot\left(-\frac{\pi}{6}\right)\right)\right) =$$

$$= 2^{2010}\left(\cos(-335\pi) + i\sin(-335\pi)\right) = 2^{2010}\left(\cos 335\pi - i\sin 335\pi\right)$$

$$= 2^{2010}\left(\cos\pi - i\sin\pi\right) = 2^{2010}(-1-0) = -2^{2010}.$$

⑥ Riješiti jednačinu $z^4 = -4$ i rješenja predstaviti u kompleksnoj ravni.

Rj. Rješenja jednačine $z^4 = -4$ su oblika

$$z_k = \sqrt[4]{|-4|}\left(\cos\frac{\varphi + 2k\pi}{4} + i\sin\frac{\varphi + 2k\pi}{4}\right), \quad k = 0,1,2,3, \quad \varphi \in [0, 2\pi)$$

$w = -4$, $|w| = \sqrt{(-4)^2 + 0^2} = 4$, $\cos\varphi = \frac{-4}{4} = -1$, $\sin\varphi = \frac{0}{4} = 0 \Rightarrow \varphi = \pi$ rad

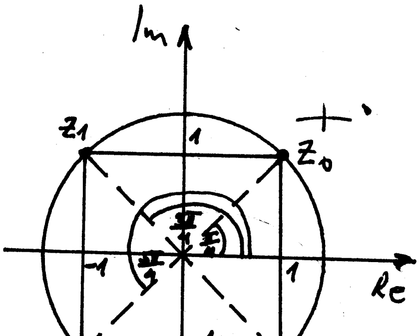
$$w = -4 = 4(\cos\pi + i\sin\pi)$$

$$z_0 = \sqrt[4]{4}\left(\cos\frac{\pi+0}{4} + i\sin\frac{\pi+0}{4}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1+i$$

$$z_1 = \sqrt[4]{4}\left[\cos\frac{\pi+2\pi}{4} + i\sin\frac{\pi+2\pi}{4}\right] = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -1+i$$

$$z_2 = \sqrt[4]{4}\left(\cos\frac{\pi+4\pi}{4} + i\sin\frac{\pi+4\pi}{4}\right) = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -1-i$$

$$z_3 = \sqrt[4]{4}\left(\cos\frac{\pi+6\pi}{4} + i\sin\frac{\pi+6\pi}{4}\right) = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = 1-i$$



Rješenja predstavljena u kompleksnoj ravni

Rješenja jednačine $z^4 = -4$ su $1+i$, $-1+i$, $-1-i$, $1-i$.

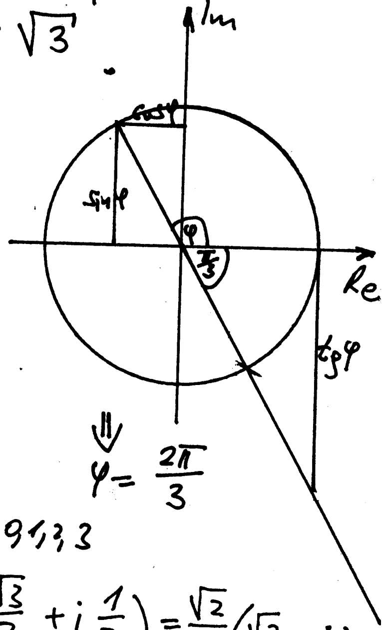
(7°) Izračunati $z = 2^{-9}(b-2)^{18}$ ako je $b = 3+2i - \frac{7-9i}{1-5i}$.

Rj. $b = 3+2i - \frac{7-9i}{1-5i} = \frac{(3+2i)(1-5i)-(7-9i)}{1-5i} = \frac{3\boxed{-15i}\boxed{+2i}+10-7\boxed{+9i}}{1-5i} = \frac{6-4i}{1-5i}\frac{(1+5i)}{(1+5i)} = \frac{26+26i}{1+25}$

$b = 1+i$, $(b-2)^2 = (i-1)^2 = -1-2i+1 = -2i$, $(b-2)^{18} = [(b-2)^2]^9 = (-2i)^9 = -2^9 \cdot i^9$

$z = 2^{-9}(b-2)^{18} = 2^{-9} \cdot (-2^9) \cdot i^8 \cdot i = (-1)(i^2)^4 \cdot i = -i$, $z = -i$
rješenje

(8°) Nađi sve vrijednosti korijena $\sqrt[4]{-2+2i\sqrt{3}}$.



$\Downarrow$

$\varphi = \frac{2\pi}{3}$

Rj. $z = \sqrt[4]{w}$

$z^4 = w$

$w = -2+2i\sqrt{3}$

$|w| = \sqrt{(-2)^2+(2\sqrt{3})^2} = \sqrt{4+12} = 4$

$\cos\varphi = \frac{-2}{4} = -\frac{1}{2}$, $\sin\varphi = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

$tg\varphi = \frac{2\sqrt{3}}{-2} = -\sqrt{3} \Rightarrow \varphi = -\frac{\pi}{3}$

$(tg\frac{\pi}{3} = \sqrt{2})$ ili $\varphi = \frac{2\pi}{3}$

$w = 4(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$

Korijeni su oblika $z_k = \sqrt[4]{|w|}(\cos\frac{\varphi+2k\pi}{4} + i\sin\frac{\varphi+2k\pi}{4})$, $k = 0,1,2,3$

$z_0 = \sqrt[4]{4}(\cos\frac{\frac{2\pi}{3}+0}{4} + i\sin\frac{\frac{2\pi}{3}+0}{4}) = \sqrt{2}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \sqrt{2}(\frac{\sqrt{3}}{2} + i\frac{1}{2}) = \frac{\sqrt{2}}{2}(\sqrt{3}+i)$

$z_1 = \sqrt[4]{4}(\cos\frac{\frac{2\pi}{3}+2\pi}{4} + i\sin\frac{\frac{2\pi}{2}+2\pi}{4}) = \sqrt{2}(\cos\frac{8\pi}{12} + i\sin\frac{8\pi}{12}) = \sqrt{2}(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$

$= \sqrt{2}(-\sin\frac{\pi}{6} + \cos\frac{\pi}{6}) = \sqrt{2}(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = \frac{\sqrt{2}}{2}(-1+i\sqrt{3})$

$z_2 = \sqrt{2}(\cos\frac{\frac{2\pi}{3}+4\pi}{4} + i\sin\frac{\frac{2\pi}{3}+4\pi}{4}) = \sqrt{2}(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}) = \sqrt{2}(-\cos\frac{\pi}{6} - i\sin\frac{\pi}{6})$

$= \sqrt{2}(-\frac{\sqrt{3}}{2} - i\frac{1}{2}) = \frac{\sqrt{2}}{2}(-\sqrt{3}-i)$

$z_3 = \sqrt{2}(\cos\frac{\frac{2\pi}{3}+6\pi}{4} + i\sin\frac{\frac{2\pi}{3}+6\pi}{4}) = \sqrt{2}(\cos\frac{20\pi}{12} + i\sin\frac{20\pi}{12}) = \sqrt{2}(\frac{1}{2} - i\frac{\sqrt{3}}{2}) = \frac{\sqrt{2}}{2}(1-i\sqrt{3})$

(9°)✓ Riješiti jednačinu $x^6 + i = \sqrt{3}$.

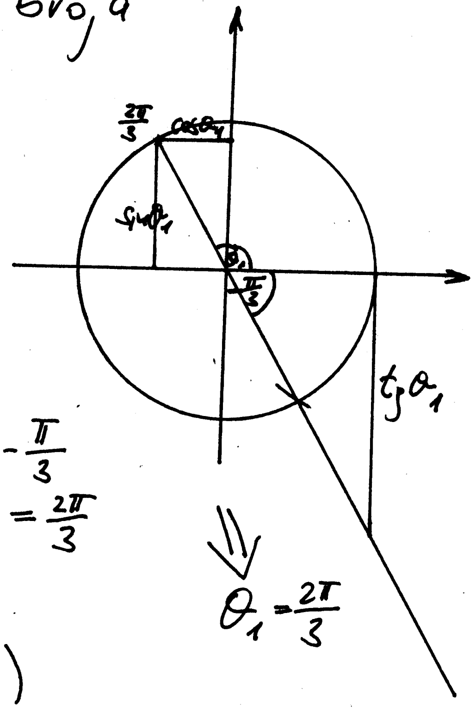
(10°)✓ Izračunati $\left(\frac{1+i}{\sqrt{3}-i}\right)^5$.

(11°)✓ Izračunati sve vrijednosti korijena $\sqrt[3]{i-1}$.

(12°)✓ Nađi sve vrijednosti $\sqrt{z}$ (ima ih 4)
ako je $z = (1+i)\sqrt{\sqrt{3}+i}$.

⑬ Odrediti realni i imaginarni dio broja

$$z=\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{17}\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$$



$\Downarrow$ $\theta_1=\frac{2\pi}{3}$

Rj. $z=z_1^{17}\cdot z_2$

$z_1=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$

$|z_1|=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$

$\sin\theta_1=\frac{\frac{\sqrt{3}}{2}}{1}=\frac{\sqrt{3}}{2}$

$\cos\theta_1=-\frac{1}{2}$

$tg\,\theta_1=-\sqrt{3}\ \xRightarrow{tg\,60^\circ=\sqrt{3}}\ \theta_1=-\frac{\pi}{3}$

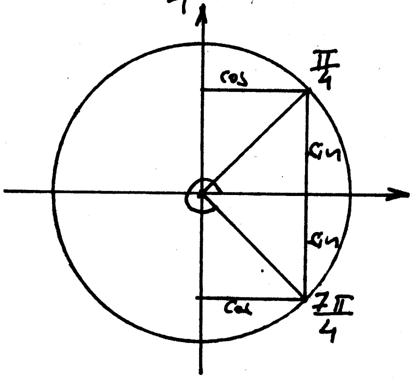
ili $\theta_1=\frac{2\pi}{3}$

$$z_1^{17}=\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^{17}=\cos 17\cdot\frac{2\pi}{3}+i\sin 17\cdot\frac{2\pi}{3}$$

$$z=z_1^{17}\cdot z_2=\left(\cos\frac{34\pi}{3}+i\sin\frac{34\pi}{3}\right)\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$$

$$=\cos\left(\frac{34\pi}{3}+\frac{5\pi}{12}\right)+i\sin\left(\frac{34\pi}{3}+\frac{5\pi}{12}\right)=\cos\frac{141\pi}{12}+i\sin\frac{141\pi}{12}$$

$$=\cos\frac{47\pi}{4}+i\sin\frac{47\pi}{4}=\cos 10\frac{7\pi}{4}+i\sin 10\frac{7\pi}{4}=\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}=$$

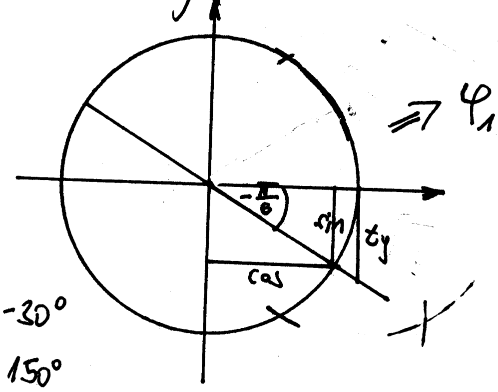


$$=\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$$

$Re(z)=\frac{\sqrt{2}}{2}$ $Im(z)=-\frac{\sqrt{2}}{2}$

realni dio broja imaginarni dio broja

⑭ Naći sve vrijednosti korijena $\sqrt[3]{z}$ ako je $z=(\sqrt{3}-i)^9$.



$\Rightarrow \varphi_1=-\frac{\pi}{6}$

Rj. $z=z_1^9$

$z_1=\sqrt{3}-i$

$|z_1|=\sqrt{3+1}=2$

$\cos\varphi_1=\frac{\sqrt{3}}{2}$

$\sin\varphi_1=-\frac{1}{2}$

$tg\,\varphi_1=-\frac{\sqrt{3}}{3}\ \xRightarrow{tg\,30^\circ=\frac{\sqrt{3}}{3}}$ $\varphi_1=-30^\circ$

ili $\varphi_1=150^\circ$

$z_1=2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$

$$z=z_1^9=2^9\left(\cos\left(-9\cdot\frac{\pi}{6}\right)+i\sin\left(-9\cdot\frac{\pi}{6}\right)\right)$$

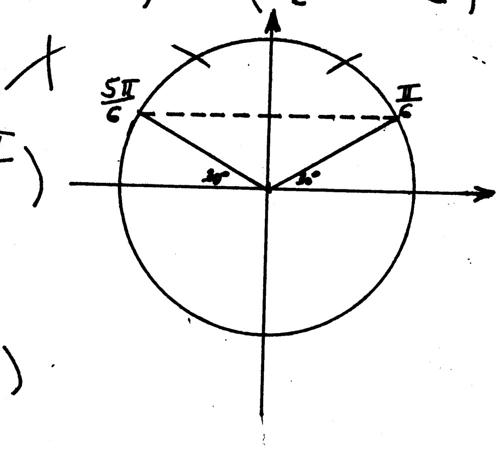
$$z=2^9\left(\cos\left(-\frac{3\pi}{2}\right)+i\sin\left(-\frac{3\pi}{2}\right)\right)=2^9\left(\cos\frac{3\pi}{2}-i\sin\frac{3\pi}{2}\right)=2^9\cdot(-i)\cdot(-1)=2^9 i$$

$$z=2^9 i=2^9\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

$\sqrt[3]{z}$ računamo po formuli $z_k=\sqrt[3]{|z|}\left(\cos\frac{\frac{\pi}{2}+2k\pi}{3}+i\sin\frac{\frac{\pi}{2}+2k\pi}{3}\right)$

$$z_0=\sqrt[3]{2^9}\left(\cos\frac{\frac{\pi}{2}}{3}+i\sin\frac{\frac{\pi}{2}}{3}\right)=\sqrt[3]{(2^3)^3}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=2^3\left(\frac{\sqrt{3}}{2}+i\cdot\frac{1}{2}\right)$$

$$=8\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)=4(\sqrt{3}+i)$$



$$z_1=8\left(\cos\frac{\frac{\pi}{2}+2\pi}{3}+i\sin\frac{\frac{\pi}{2}+2\pi}{3}\right)=8\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$$

$$=8\left(-\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=8\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)=4(-\sqrt{3}+i)$$

$$z_2=8\left(\cos\frac{\frac{\pi}{2}+4\pi}{3}+i\sin\frac{\frac{\pi}{2}+4\pi}{3}\right)=8\left(\cos\frac{9\pi}{6}+i\sin\frac{9\pi}{6}\right)$$

$$=8\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)=8(0+i(-1))=-8i$$

Vrijednosti $\sqrt[3]{z}$ su $4(\sqrt{3}+i)$, $4(-\sqrt{3}+i)$ i $-8i$.

(15.) Nađi sve vrijednasti $\sqrt[3]{z}$ ako je $z=(\sqrt{3}-i)^5(1+i\sqrt{3})$.

Rj. $z=z_1^5\cdot z_2=(\sqrt{3}-i)^5\cdot(1+i\sqrt{3})=(\sqrt{3}-i)^4\cdot(\sqrt{3}-i)(1+i\sqrt{3})$

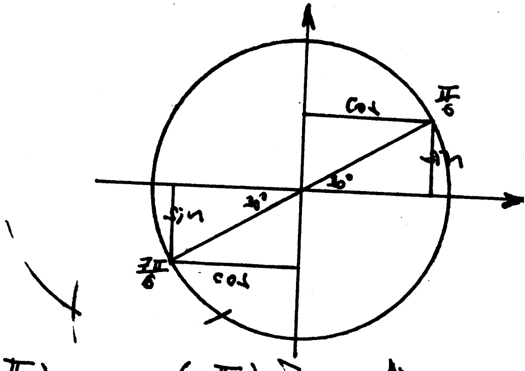
$$(\sqrt{3}-i)^2=3-2i\sqrt{3}+i^2=2-2\sqrt{3}$$

$$(\sqrt{3}-i)^4=(2-2\sqrt{3})^2=4-8i\sqrt{3}+4\cdot 3i^2=-8-8i\sqrt{3}$$

$$(\sqrt{3}-i)(1+i\sqrt{3})=\sqrt{3}+3i-i-i^2\sqrt{3}=2\sqrt{3}+2i$$

$$z=(-8-8i\sqrt{3})(2\sqrt{3}+2i)=-16\sqrt{3}-16i-48i+16\sqrt{3}=-64i=-2^6 i$$

$$z=2^6\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]$$



$$z_k=\sqrt[3]{|z|}\left(\cos\frac{\theta+2k\pi}{3}+i\sin\frac{\theta+2k\pi}{3}\right)$$

$$z_0=\sqrt[3]{2^6}\left(\cos\frac{-\frac{\pi}{2}}{3}+i\sin\frac{-\frac{\pi}{2}}{3}\right)=\sqrt[3]{(2^2)^3}\left[\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right]$$

$$=4\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)=4\left(\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)=2(\sqrt{3}-i)$$

$$z_1=4\left(\cos\frac{-\frac{\pi}{2}+2\pi}{3}+i\sin\frac{-\frac{\pi}{2}+2\pi}{3}\right)=4\left(\cos\frac{\frac{3\pi}{2}}{3}+i\sin\frac{\frac{3\pi}{2}}{3}\right)=4\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

$$=4(0+i)=4i$$

$$z_2=4\left(\cos\frac{-\frac{\pi}{2}+4\pi}{3}+i\sin\frac{-\frac{\pi}{2}+4\pi}{3}\right)=4\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)=4\left(-\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)=4\left(-\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)$$

Tražena rješenja su $\sqrt[3]{z}\in\{2(\sqrt{3}-i),\ 4i,\ -2(\sqrt{3}+i)\}$

16. Riješiti jednačinu $\dfrac{(3+2i)(1+i)+2i}{(2-i)(1+i)-3} = \dfrac{7-i}{-4}\cdot z^4$.

Rj: $(3+2i)(1+i)+2i = \underline{3}+3i+2i+\underline{2i^2}+2i = 1+7i$

$(2-i)(1+i)-3 = \underline{2}+2i-i\underline{-i^2}\underline{-3} = i$

Sad jednačina $\dfrac{(3+2i)(1+i)+2i}{(2-i)(1+i)-3} = \dfrac{7-i}{-4}\cdot z^4$ postaje

$\dfrac{1+7i}{i} = \dfrac{7-i}{-4}\cdot z^4$, kako je $\dfrac{1+7i}{i}\cdot\dfrac{i}{i} = \dfrac{i+7i^2}{i^2} = \dfrac{-7+i}{-1} = 7-i$

imamo $7-i = \dfrac{7-i}{-4}\cdot z^4 \quad /\cdot\dfrac{1}{7-i}$

$1 = \dfrac{1}{-4}\cdot z^4 \Rightarrow z^4 = -4$ primjetite da smo ovu jednačinu riješili u zadatku broj 6.

17. Napisati u trigonometriskom i eksponencijalnom obliku kompleksni broj $z = 2\sqrt{3}+2i$, a zatim naći $\sqrt[4]{z}$.

18. Napisati u trigonometriskom i eksponencijalnom obliku kompleksni broj $z = \dfrac{-1-i}{2}$, a zatim naći z^{14}.

19. Izračunati $z = 2^{-6}(a-2i)^{18}$, ako je $a = \dfrac{8+i}{3+2i} - 3 + 2i$.

20. Izračunati broj $z = \dfrac{\left(\frac{1}{2\sqrt{3}} - \frac{i}{2}\right)^9}{\left(-1+\frac{i}{\sqrt{3}}\right)^6}$.

21. Izračunati $\left(\dfrac{1+i\sqrt{3}}{2}\right)^{60} + \left(\dfrac{1-i\sqrt{3}}{2}\right)^{30}$.

22. Odrediti prirodan broj x iz uslova $(3+4i)^{x-1} - (1+i)^4 = 5^x$.

Ⓣ Napisati sva rješenja jednačine $x^4+x^2+1=0$ u trigonometriskom obliku.

Rj uvodimo smjenu $x^2=t$

$$t^2+t+1=0$$

$$D=1-4=-3=3i^2$$

$$t_{1,2}=\frac{-1\pm i\sqrt{3}}{2}$$

$$t_1=\frac{-1-i\sqrt{3}}{2}=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

$$t_2=\frac{-1+i\sqrt{3}}{2}=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

$$t_1=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$$

$$t_2=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$$

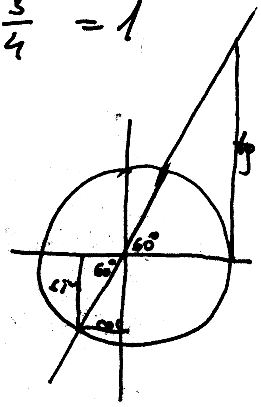
$$|t_1|=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$$

$$\cos\varphi_1=-\frac{1}{2}$$

$$\sin\varphi_1=-\frac{\sqrt{3}}{2}$$

$$tg\,\varphi_1=\sqrt{3}$$

$$tg\,60°=\sqrt{3}$$



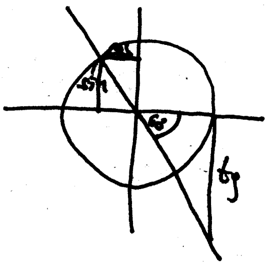
$$\varphi_1=240°=\frac{4\pi}{3}$$

$$|t_2|=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$$

$$\cos\varphi_2=-\frac{1}{2}$$

$$\sin\varphi_2=\frac{\sqrt{3}}{2}$$

$$tg\,\varphi_2=-\sqrt{3}$$



$$\varphi_2=120°=\frac{2\pi}{3}$$

$$x^2=t$$

$$z=\sqrt{t_1},\quad z_k=\sqrt{|t_1|}\left(\cos\frac{\varphi+2k\pi}{2}+i\sin\frac{\varphi+2k\pi}{2}\right),\quad k=0,1$$

$$z_0=\sqrt{1}\left(\cos\frac{\frac{4\pi}{3}}{2}+i\sin\frac{\frac{4\pi}{3}}{2}\right)=\cos\frac{4\pi}{6}+i\sin\frac{4\pi}{6}=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$$

$$z_1=\sqrt{1}\left(\cos\frac{\frac{4\pi}{3}+2\pi}{2}+i\sin\frac{\frac{4\pi}{3}+2\pi}{2}\right)=\cos\frac{10\pi}{6}+i\sin\frac{10\pi}{6}=\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$$

$$z=\sqrt{t_2}$$

$$z_0=\sqrt{1}\left(\cos\frac{\frac{2\pi}{3}}{2}+i\sin\frac{\frac{2\pi}{3}}{2}\right)=\cos\frac{2\pi}{6}+i\sin\frac{2\pi}{6}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

$$z_1=\sqrt{1}\left(\cos\frac{\frac{2\pi}{3}+2\pi}{2}+i\sin\frac{\frac{2\pi}{3}+2\pi}{2}\right)=\cos\frac{8\pi}{6}+i\sin\frac{8\pi}{6}=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}.$$

Sva rješenja jednačine $x^4+x^2+1=0$ napisana u trigonometriskom obliku su:

$x_1=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$, $x_2=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$, $x_3=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$

i $x_4=\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$.

(#) Riješiti jednačinu $x^4+\frac{9}{4}=0$ i rješenja predstaviti u kompleksnoj ravni.

Rj. $x^4=-\frac{9}{4}$

$x=\sqrt[4]{-\frac{9}{4}}$

$x=\sqrt[4]{z}$

n-ti korijen kompleksnog broja tražimo po formuli:

$$z_k=\sqrt[n]{|z|}\left(\cos\frac{\omega+2k\pi}{n}+i\sin\frac{\omega+2k\pi}{n}\right),\quad k=1,2,\dots,n$$

$z=-\frac{9}{4}$ $\quad |z|=\sqrt{\left(\frac{9}{4}\right)^2+0^2}=\frac{9}{4}$ $\qquad z=\frac{9}{4}(\cos\pi+i\sin\pi)$

$$\cos\omega=\frac{9}{|z|}=\frac{-\frac{9}{4}}{\frac{9}{4}}=-1$$

$$\Rightarrow \omega=\pi$$

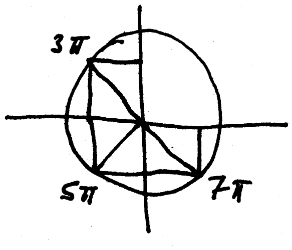
$$\sin\omega=\frac{0}{|z|}=0$$

$$z_0=\sqrt[4]{\frac{9}{4}}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt[4]{\left(\frac{3}{2}\right)^2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\frac{\sqrt{3}}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)$$

$$z_0=\frac{\sqrt{3}}{2}+i\frac{\sqrt{3}}{2}$$

$$z_1=\sqrt{\frac{3}{2}}\left(\cos\frac{\pi+2\pi}{4}+i\sin\frac{\pi+2\pi}{4}\right)=\frac{\sqrt{3}}{\sqrt{2}}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=\frac{\sqrt{3}}{\sqrt{2}}\left(-\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

$$=\frac{\sqrt{3}}{\sqrt{2}}\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)=-\frac{\sqrt{3}}{2}+i\frac{\sqrt{3}}{2}$$

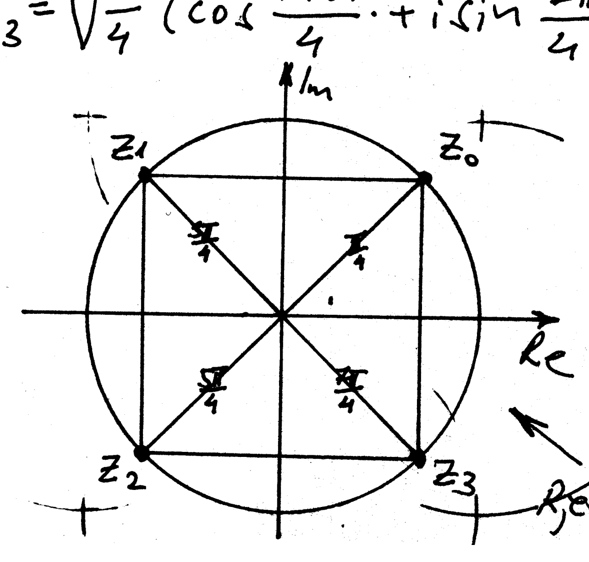


$$z_2=\sqrt[4]{\frac{9}{4}}\left(\cos\frac{\pi+4\pi}{4}+i\sin\frac{5\pi}{4}\right)=\frac{\sqrt{3}}{\sqrt{2}}\left(-\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)$$

$$=\frac{\sqrt{3}}{\sqrt{2}}\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)=-\frac{\sqrt{3}}{2}-i\frac{\sqrt{3}}{2}$$

$$z_3=\sqrt[4]{\frac{9}{4}}\left(\cos\frac{\pi+6\pi}{4}+i\sin\frac{7\pi}{4}\right)=\frac{\sqrt{3}}{\sqrt{2}}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)=\frac{\sqrt{3}}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)$$

$$=\frac{\sqrt{3}}{2}-i\frac{\sqrt{3}}{2}$$



Rješenja jednačine su:

$\frac{\sqrt{3}}{2}+i\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{2}+i\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{2}-i\frac{\sqrt{3}}{2}$

i $\frac{\sqrt{3}}{2}-i\frac{\sqrt{3}}{2}$

Rješenja predstavljena u kompleksnoj ravni.

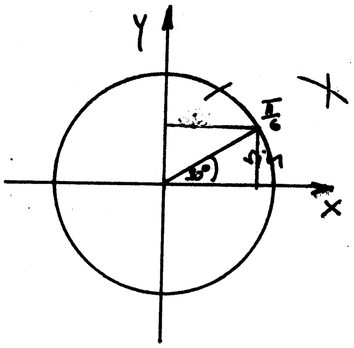
(#) Izračunati $\frac{(\sqrt{3}+i)^{22}(1-i)^{15}}{(-1-i)^{3}}$.

Rj: $z_1=\sqrt{3}+i$

$|z_1|=\sqrt{3+1}=\sqrt{4}=2$

$\cos\theta_1=\frac{a}{|z_1|}=\frac{\sqrt{3}}{2}$

$\sin\theta_1=\frac{b}{|z_1|}=\frac{1}{2}$



$\sin 30°=\sin\frac{\pi}{6}=\frac{1}{2}$

$\theta_1=\frac{\pi}{6}$

$z_1=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$z_1^{22}=2^{22}\left(\cos\left(22\cdot\frac{\pi}{6}\right)+i\sin\left(22\cdot\frac{\pi}{6}\right)\right)$

$=2^{22}\left(\cos\frac{11\pi}{3}+i\sin\frac{11\pi}{3}\right)$

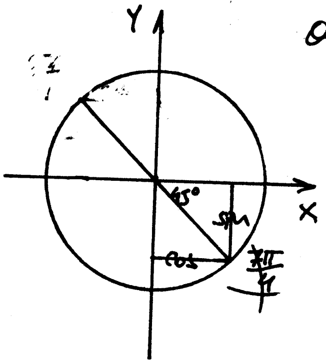
$z_2=1-i$

$|z_2|=\sqrt{1+1}=\sqrt{2}$

$\cos\theta_2=\frac{a}{|z_2|}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$

$\sin\theta_2=\frac{b}{|z_2|}=\frac{-1}{\sqrt{2}}=-\frac{\sqrt{2}}{2}$

$tg\,\theta_2=\frac{b}{a}=-1$



$tg\,45°=tg\,\frac{\pi}{4}=1$

$\theta_2=\frac{7\pi}{4}$

$z_2=\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$

$z_2^{15}=(\sqrt{2})^{15}\left(\cos 15\cdot\frac{7\pi}{4}+i\sin 15\frac{7\pi}{4}\right)$

$=2^{7}\sqrt{2}\left(\cos\frac{105\pi}{4}+i\sin\frac{105\pi}{4}\right)$

$(1-i)^2=1-2i+i^2=-2i$

$(1-i)^{14}=((1-i)^2)^7=(-2i)^7=-2^7\cdot i^7=-2^7\cdot i^6\cdot i$

$=-2^7\cdot(i^2)^3\cdot i=2^7\cdot i$

$z_3=-1-i$

$(-1-i)^2=1+2i+i^2=1+2i-1=2i$

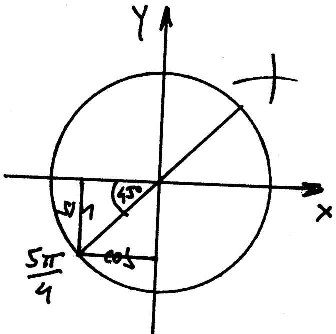
$(-1-i)^3=(-1-i)^2\cdot(-1-i)=2i(-1-i)=-2i-2i^2=2-2i$

$|z_3|=\sqrt{1+1}=\sqrt{2}$

$\cos\theta_3=\frac{a}{|z_3|}=\frac{-1}{\sqrt{2}}=-\frac{\sqrt{2}}{2}$

$\sin\theta_3=\frac{b}{|z_3|}=\frac{-1}{\sqrt{2}}=-\frac{\sqrt{2}}{2}$

$tg\,\theta_2=\frac{b}{a}=\frac{-1}{-1}=1$



$tg\,45°=tg\,\frac{\pi}{4}=1$

$z_3=\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$

$z_3^{3}=(\sqrt{2})^{3}\left(\cos 3\cdot\frac{5\pi}{4}+i\sin 3\cdot\frac{5\pi}{4}\right)$

$=2\sqrt{2}\left(\cos\frac{15\pi}{4}+i\sin\frac{15\pi}{4}\right)$

$$\frac{(1-i)^{15}}{(-1-i)^3}=\frac{2^7\cancel{\sqrt{2}}}{2\cancel{\sqrt{2}}}\left(\cos\frac{105\pi-15\pi}{4}+i\sin\frac{105-15}{4}\pi\right)=2^6\left(\cos\frac{90\pi}{4}+i\sin\frac{90\pi}{4}\right)$$

$$z_1^{22}\cdot\frac{z_2^{15}}{z_3^{3}}=2^{22}\cdot2^{6}\left(\cos\left(\frac{11\pi}{3}+\frac{90\pi}{4}\right)+i\sin\left(\frac{11\pi}{3}+\frac{90\pi}{4}\right)\right)=2^{28}\left(\cos\frac{314}{12}\pi+i\sin\frac{157}{6}\pi\right)$$

$$z=2^{28}\left(\cos\left(\frac{\pi}{6}+2\cdot13\pi\right)+i\sin\left(\frac{\pi}{6}+2\cdot13\pi\right)\right)=2^{28}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=2^{28}\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)=2^{27}(\sqrt{3}+i)$$

(#) Naći sve vrijednosti korijena $\sqrt[6]{-27}$.

Rj. Označimo sa $z=\sqrt[6]{-27}$

$$z^6=-27$$

Teorema Jednačina $z^n=w$, gdje je w po volji odabran kompleksan broj različit od 0 ima tačno n različitih rješenja koji su oblika

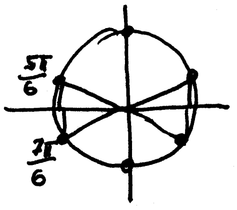
$$z_k=\sqrt[n]{|w|}\left(\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right)$$

gdje je φ najmanji pozitivan ugao iz intervala $[0,2\pi)$ takav da $w=|w|(\cos\varphi+i\sin\varphi)$, a $k=0,1,2,\dots,n-1$.

U našem slučaju $w=-27 \Rightarrow |w|=\sqrt{(-27)^2+0^2}=27$

$w=a+ib$ $\qquad$ $\left(|w|=\sqrt{a^2+b^2}\right)$

$$\left.\begin{aligned}\cos\varphi&=\frac{-27}{27}\left(=\frac{a}{|w|}\right)=-1\\ \sin\varphi&=\frac{b}{|w|}=\frac{0}{27}=0\end{aligned}\right\}\Rightarrow \varphi=\pi$$



$$w=-27=27(\cos\pi+i\sin\pi)$$

$$z_0=\sqrt[6]{27}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=\left(3^3\right)^{\frac{1}{6}}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=\sqrt{3}\left(\frac{\sqrt{3}}{2}+i\,\frac{1}{2}\right)$$

$$z_1=\sqrt[6]{27}\left(\cos\frac{\pi+2\pi}{6}+i\sin\frac{\pi+2\pi}{6}\right)=\sqrt{3}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=i\sqrt{3}$$

$$z_2=\sqrt[6]{27}\left(\cos\frac{\pi+4\pi}{6}+i\sin\frac{\pi+4\pi}{6}\right)=\sqrt{3}\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)=\sqrt{3}\left(-\frac{\sqrt{3}}{2}+i\,\frac{1}{2}\right)$$

$$z_3=\sqrt[6]{27}\left(\cos\frac{\pi+6\pi}{6}+i\sin\frac{\pi+6\pi}{6}\right)=\sqrt{3}\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)=\sqrt{3}\left(-\frac{\sqrt{3}}{2}-i\,\frac{1}{2}\right)$$

$$z_4=\sqrt[6]{27}\left(\cos\frac{\pi+8\pi}{6}+i\sin\frac{\pi+8\pi}{6}\right)=\sqrt{3}\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)=-i\sqrt{3}$$

$$z_5=\sqrt[6]{27}\left(\cos\frac{\pi+10\pi}{6}+i\sin\frac{11\pi}{6}\right)=\sqrt{3}\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)=\sqrt{3}\left(\frac{\sqrt{3}}{2}-i\,\frac{1}{2}\right)$$

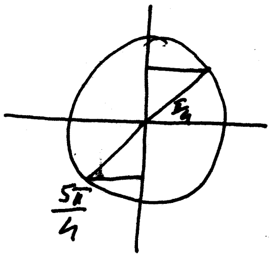
Sve vrijednosti korijena $\sqrt[6]{-27}$ su: $\frac{3}{2}+i\frac{\sqrt{3}}{2}$, $i\sqrt{3}$, $-\frac{3}{2}+i\,\frac{\sqrt{3}}{2}$, $-\frac{3}{2}-i\,\frac{\sqrt{3}}{2}$, $-i\sqrt{3}$ i $\frac{3}{2}-i\,\frac{\sqrt{3}}{2}$.

(#) Ako je $z=\frac{1}{2}-i\frac{\sqrt{3}}{2}$, izračunati sve vrijednosti korijena $\sqrt[3]{(z+\frac{1}{z}+i)^5}$.

Rj. $z=\frac{1}{2}-i\frac{\sqrt{3}}{2}$, $z+\frac{1}{z}=\frac{1}{2}-i\frac{\sqrt{3}}{2}+\frac{1}{\frac{1-i\sqrt{3}}{2}}=\frac{1}{2}-i\frac{\sqrt{3}}{2}+\frac{2}{1-i\sqrt{3}}\cdot\frac{(1+i\sqrt{3})}{(1+i\sqrt{3})}$

$$=\frac{1}{2}-i\frac{\sqrt{3}}{2}+\frac{2+2i\sqrt{3}}{1+3}=\frac{1}{2}-i\cancel{\frac{\sqrt{3}}{2}}+\frac{1}{2}+i\cancel{\frac{\sqrt{3}}{2}}=1$$

$z+\frac{1}{z}+i=1+i$ Uvedimo oznaku $w=z+\frac{1}{z}+i=1+i$



$|w|=\sqrt{2}$

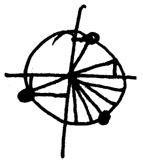
$$\left.\begin{aligned}\cos\varphi&=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}\\ \sin\varphi&=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}\\ \operatorname{tg}\varphi&=1\end{aligned}\right\}\Rightarrow \varphi=45^\circ=\frac{\pi}{4}\text{ rad}$$

$w=\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$

$w^5=(\sqrt{2})^5(\cos 5\cdot\frac{\pi}{4}+i\sin 5\cdot\frac{\pi}{4})=4\sqrt{2}(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4})$

$w^n=c$ gdje je c kompleksan broj ima tačno n rješenja
$w_k=\sqrt[n]{|c|}(\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n})$, φ najmanji pozitivan ugao iz $[0,2\pi)$
$k=1,2,\dots,n$

Mi treba da nađemo $\sqrt[3]{(z+\frac{1}{z}+i)^5}$ tj. $\sqrt[3]{w^5}$



$V_1=\sqrt[3]{4\sqrt{2}}(\cos\frac{\frac{5\pi}{4}+0}{3}+i\sin\frac{5\pi}{12})=32^{\frac{1}{6}}(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12})=$
$=\sqrt[6]{32}(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12})$

$V_2=\sqrt[6]{32}(\cos\frac{\frac{5\pi}{4}+2\pi}{3}+i\sin\frac{\frac{5\pi+8\pi}{4}}{3})=\sqrt[6]{32}(\cos\frac{13\pi}{12}+i\sin\frac{13\pi}{12})$

$V_3=\sqrt[6]{32}(\cos\frac{\frac{5\pi}{4}+4\pi}{3}+i\sin\frac{\frac{5\pi}{4}+\frac{16\pi}{4}}{3})=\sqrt[6]{32}(\cos\frac{21\pi}{12}+i\sin\frac{21\pi}{12})$

Napišimo rješenja v_1, v_2 i v_3 u obliku $a+ib$:

$\cos\frac{\pi}{12}=\cos(\frac{\pi}{3}-\frac{\pi}{4})=\cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}=\frac{1}{2}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{2}+\sqrt{6}}{4}$

$\sin\frac{\pi}{12}=\sin(\frac{\pi}{3}-\frac{\pi}{4})=\sin\frac{\pi}{3}\cos\frac{\pi}{4}-\cos\frac{\pi}{3}\sin\frac{\pi}{4}=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$

Kako je $\cos\frac{5\pi}{12} = \sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$, $\sin\frac{5\pi}{12} = \cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$

to je $v_1 = \sqrt[6]{32}\left(\frac{\sqrt{6}-\sqrt{2}}{4} + i\,\frac{\sqrt{6}+\sqrt{2}}{4}\right)$

$\cos\frac{13\pi}{12} = -\cos\frac{\pi}{12} = -\frac{\sqrt{6}+\sqrt{2}}{4}$, $\sin\frac{13\pi}{12} = -\sin\frac{\pi}{12} = -\frac{\sqrt{6}-\sqrt{2}}{4}$

$v_2 = \sqrt[6]{32}\left(-\frac{\sqrt{6}+\sqrt{2}}{4} - i\,\frac{\sqrt{6}-\sqrt{2}}{4}\right)$

$\cos\frac{21\pi}{12} = \cos\frac{7\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\sin\frac{21\pi}{4} = \sin\frac{7\pi}{4} = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

$v_3 = \sqrt[6]{32}\left(\frac{\sqrt{2}}{2} - i\,\frac{\sqrt{2}}{2}\right)$

v_1, v_2 i v_3 su tražena rješenja

Nađi sve vrijednosti korijena $\sqrt[4]{z}$, ako je $z=(-1+i)^8$.

Rj. $\sqrt[4]{z}$, $z = z_1^8$, $z_1 = -1+i$, $|z_1| = \sqrt{2}$

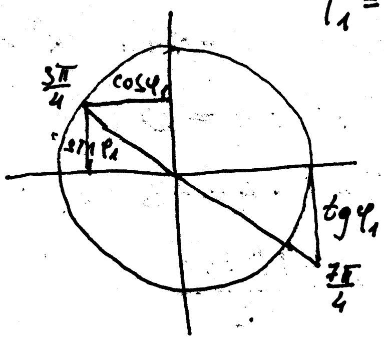
tg 45° = 1

$\varphi_1 = \frac{3\pi}{4}$

$\cos\varphi_1 = \frac{-1}{\sqrt{2}}$

$\sin\varphi_1 = \frac{1}{\sqrt{2}}$

$\mathrm{tg}\,\varphi_1 = \frac{1}{-1} = -1$



$z_1 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$z = z_1^8 = (\sqrt{2})^8\left[\cos 8\cdot\frac{3\pi}{4} + i\sin 8\cdot\frac{3\pi}{4}\right]$

$z = 16(\cos 6\pi + i\sin 6\pi) = 16(\cos 0 + i\sin 0)$

$\sqrt[4]{z} = ?$ $\quad z_k = \sqrt[4]{|z|}\left(\cos\frac{0+2k\pi}{4} + i\sin\frac{0+2k\pi}{4}\right)$

$z_0 = \sqrt[4]{16}\left(\cos\frac{0}{4} + i\sin\frac{0}{4}\right) = 2(1 + i\cdot 0) = 2$

$z_1 = \sqrt[4]{16}\left(\cos\frac{0+2\pi}{4} + i\sin\frac{0+2\pi}{4}\right) = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2(0 + i\cdot 1) = 2i$

$z_2 = \sqrt[4]{16}\left(\cos\frac{0+4\pi}{4} + i\sin\frac{0+4\pi}{4}\right) = 2(\cos\pi + i\sin\pi) = 2(-1 + i\cdot 0) = -2$

$z_3 = \sqrt[4]{16}\left(\cos\frac{0+6\pi}{4} + i\sin\frac{0+6\pi}{4}\right) = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = 2(0 + i\cdot(-1)) = -2i$

Sve vrijednosti $\sqrt[4]{z}$ su $\{2, 2i, -2, -2i\}$

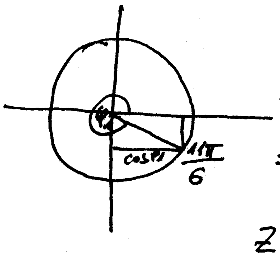
\# Izračunati $\left(1-\frac{\sqrt{3}-i}{2}\right)^{24}(2+\sqrt{3})^{12}$.

Rj. Označimo sa $z_1=\sqrt{3}-i$. Tada $|z_1|=\sqrt{3+1}=2$

$\cos\varphi_1=\frac{\sqrt{3}}{2}\quad\left(=\frac{a}{|z|}\right)$

$\sin\varphi_1=-\frac{1}{2}\quad\left(=\frac{b}{|z|}\right)$

$tg\varphi_1=\frac{b}{a}=-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$



$tg\frac{\pi}{6}=30°$

$\Rightarrow \varphi_1=\frac{11\pi}{6}=-\frac{\pi}{6}$

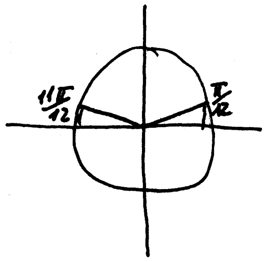
$z_1=2\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$

$\left(1-\frac{z_1}{2}\right)=\left(1-\cos\frac{11\pi}{6}-i\sin\frac{11\pi}{6}\right)$ Znamo da je $\cos 2x=\cos^2x-\sin^2x$

$\sin 2x=2\sin x\cos x$

$$\left.\begin{array}{l}1=\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\\ \underline{\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}\end{array}\right\}\Rightarrow\quad 1-\cos x=2\sin^2\frac{x}{2}$$

$1-\cos\frac{11\pi}{6}=2\sin^2\frac{11\pi}{12}$



$\sin\frac{11\pi}{6}=2\sin\frac{11\pi}{12}\cos\frac{11\pi}{12}$

$$\left(1-\frac{1}{2}z_1\right)=\left(1-\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)=\left(2\sin^2\frac{11\pi}{12}-2i\sin\frac{11\pi}{12}\cos\frac{11\pi}{12}\right)=$$

$$=2\sin\frac{11\pi}{12}\left(\sin\frac{11\pi}{12}-i\cos\frac{11\pi}{12}\right)=2i\sin\frac{11\pi}{12}\left(-\cos\frac{11\pi}{12}-i\sin\frac{11\pi}{12}\right)=$$

$$=-2i\sin\frac{11\pi}{12}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)$$

$$\sin\frac{11\pi}{12}=\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\sin\frac{\pi}{4}\cos\frac{\pi}{6}-\sin\frac{\pi}{6}\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=$$

$$=\frac{\sqrt{6}-\sqrt{2}}{4}=\frac{\sqrt{2}(\sqrt{3}-1)}{4},\quad(\sqrt{3}-1)^2=3-2\sqrt{3}+1=4-2\sqrt{3}=2(2-\sqrt{3})$$

$$\sin^2\frac{11\pi}{12}=\sin^2\frac{\pi}{12}=\frac{2(\sqrt{3}-1)^2}{16}=\frac{2(2-\sqrt{3})}{8}=\frac{2-\sqrt{3}}{4},\qquad i^{24}=(i^2)^{12}=(-1)^{12}=1$$

$$\left(1-\frac{\sqrt{3}-i}{2}\right)^{24}(2+\sqrt{3})^{12}=(-2i)^{24}\left(\sin\frac{11\pi}{12}\right)^{24}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)^{24}\cdot(2+\sqrt{3})^{12}$$

$$=(-2)^{24}\left(\sin^2\frac{11\pi}{12}\right)^{12}\left(\cos 24\cdot\frac{11\pi}{12}+i\sin 24\cdot\frac{11\pi}{12}\right)\cdot(2+\sqrt{3})^{12}=\cancel{2^{24}}\cdot\frac{(2-\sqrt{3})^{12}}{\cancel{2^{24}}}\cdot$$

$$\cdot(\cos 22\pi+i\sin 22\pi)\cdot(2+\sqrt{3})^{12}=(4-3)^{12}\cdot 1=1$$ traženo rješenje

(#) Riješiti jednačinu u skupu kompleksnih brojeva:

$$(2+5i)\,z^3-2i+5=0\,.$$

Rj.

$$(2+5i)\,z^3-2i+5=0$$

$$(2+5i)\,z^3=2i-5$$

$$z^3=\frac{(2i-5)\cdot(2-5i)}{(2+5i)\cdot(2-5i)}=\frac{4i-10i^2-10+25i}{4-25i^2}=\frac{29i}{29}$$

$$z^3=i$$

$$z=\sqrt[3]{i}$$

Jednačina $z^n=w$ gdje je w kompleksan broj ima n rješenja koje tražimo u obliku

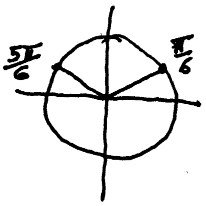
$$z_k=\sqrt[n]{|w|}\left(\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right)$$

U našem slučaju $w=i$, $w=a+bi$ $\qquad k=0,1,\ldots,k-1$

$$|w|=\sqrt{a^2+b^2}=\sqrt{1}=1$$

$$\cos\varphi=\frac{a}{|z|}=0,\quad \sin\varphi=\frac{b}{|z|}=\frac{1}{1}=1 \quad\Rightarrow\quad \varphi=\frac{\pi}{2}$$

$$i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$$



$$z_0=1\cdot\left(\cos\frac{\frac{\pi}{2}}{3}+i\sin\frac{\pi}{6}\right)=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

$$z_1=1\left(\cos\frac{\frac{\pi}{2}+2\pi}{3}+i\sin\frac{\pi/2+2\pi}{3}\right)=\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

$$z_2=1\left(\cos\frac{\frac{\pi}{2}+4\pi}{3}+i\sin\frac{\pi/2+4\pi}{3}\right)=\cos\frac{9\pi}{6}+i\sin\frac{3\pi}{2}=-i$$

Rješenja jednačine u skupu kompleksnih brojeva su $z_0=\frac{\sqrt{3}}{2}+\frac{1}{2}i$, $z_1=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$ i $z_2=-i$.

(#) Dokazati da je proizvod svih n-tih korijena iz 1 jednak $(-1)^{n-1}$ $(1 \in \mathbb{C})$.

$z = a + ib$
$z = |z|(\cos\varphi + i\sin\varphi)$

Rj. $1 = \cos 0^\circ + i\sin 0^\circ$, $\cos 0^\circ = 1$, $\sin 0^\circ = 0$

$z = 1$, $|z| = \sqrt{a^2+b^2} = 1$, $\varphi = 0$

$\sqrt[n]{1}$ ima n rješenja

$$z_k = \sqrt[n]{|z|}\left(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi + 2k\pi}{n}\right), \quad k = 0,1,2,\dots,n-1$$

u našem slučaju $|z| = 1$, $\varphi = 0$ pa imamo

$$z_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad k = 0,1,2,\dots,n-1$$

Kako množimo dva kompleksna broja.

$z_1 = |z_1|(\cos\varphi_1 + i\sin\varphi_1)$
$z_2 = |z_2|(\cos\varphi_2 + i\sin\varphi_2)$

$z_1 \cdot z_2 = |z_1||z_2|\left(\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)\right)$

U našem slučaju

$$z_0 \cdot z_1 \cdot z_2 \cdot \ldots \cdot z_{n-1} = (\cos 0^\circ + i\sin 0^\circ)\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)\left(\cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}\right)\cdot$$

$$\ldots\cdot\left(\cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}\right) =$$

$$= \cos\frac{1}{n}\left(2\pi + 4\pi + \ldots + 2(n-1)\pi\right) + i\sin\frac{1}{n}\left(2\pi + 4\pi + \ldots + 2(n-1)\pi\right) \overset{(*)}{=}$$

Kako sabrati $2 + 4 + 6 + \ldots + 2(n-1)$?

$$S = 2 + 4 + 6 + \quad + 2(n-1)$$
$$S = 2(n-1) + 2(n-2) + 2(n-3) + \ldots + 2$$

$$2S = \underbrace{2(n-1)+2}_{2n} + \underbrace{2(n-2)+4}_{2n} + \underbrace{2(n-3)+6}_{2n} + \ldots + \underbrace{2(n-1)+2}_{2n}$$

$$2S = (n-1)\cdot 2n \Rightarrow S = (n-1)\cdot n$$

$$\overset{(*)}{=} \cos\frac{1}{n}\cdot(n-1)\cdot n\cdot\pi + i\sin\frac{1}{n}(n-1)\cdot n\cdot\pi = \cos(n-1)\pi + \overbrace{i\sin(n-1)\pi}^{=0\ \forall n}$$

$= (-1)^{n-1}$ što je i trebalo dobiti

(#) Izračunati $(\sqrt{3}-i)^{2002}$. Rezultat predstaviti u algebarskom obliku.

$z=|z|(\cos\theta+i\sin\theta)$

Rj.

$z=\sqrt{3}-i$

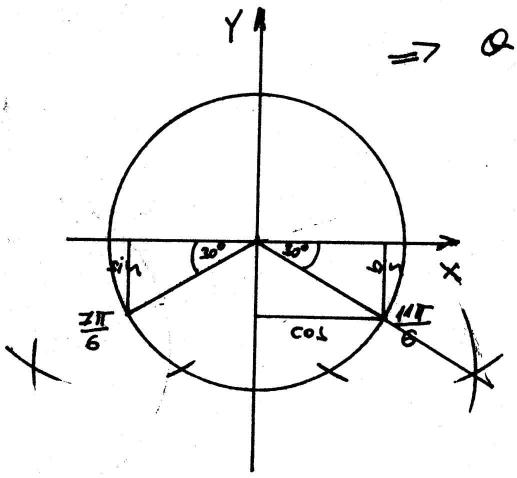
$|z|=\sqrt{3+1}=\sqrt{4}=2$

$\cos\theta=\frac{a}{|z|}=\frac{\sqrt{3}}{|z|}=\frac{\sqrt{3}}{2}$

$\sin\theta=\frac{b}{|z|}=\frac{-1}{2}$

$tg\,\theta=\frac{b}{a}=-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$

$\sin 30^\circ=\sin\frac{\pi}{6}=\frac{1}{2}$



$\Rightarrow \theta=\frac{11\pi}{6}$

$z=\sqrt{3}-i=$

$=2\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$

$z^n=|z|^n(\cos n\cdot\theta+i\sin n\cdot\theta)$

$z=2^{2002}\left(\cos\left(\overset{1001}{\cancel{2002}}\cdot\frac{11\pi}{\underset{3}{\cancel{6}}}\right)+i\sin\left(\overset{1001}{\cancel{2002}}\cdot\frac{11\pi}{\underset{3}{\cancel{6}}}\right)\right)=$

$=2^{2002}\left(\cos\frac{11\,011}{3}\pi+i\sin\frac{11\,011}{3}\pi\right)=2^{2002}\left(\cos\left(3670\pi+\frac{\pi}{3}\right)\right.$

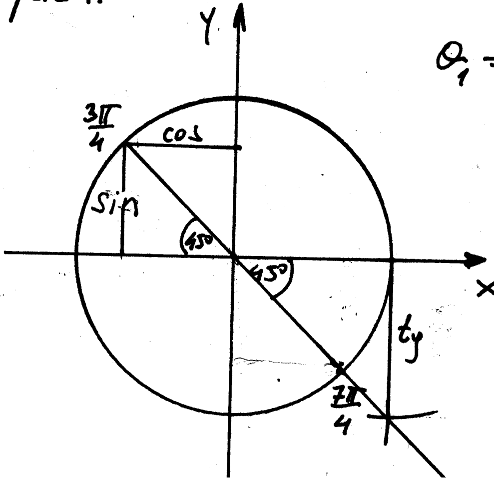
$\left.i\sin\left(3670\pi+\frac{\pi}{3}\right)\right)=2^{2002}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=2^{2002}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$

$z^{2002}=2^{2001}(1+i\sqrt{3})$

$(\sqrt{3}-i)^{2002}=2^{2001}(1+i\sqrt{3})$

(#)^{v} Kompleksan broj $z=\frac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$ napisati u trigonometriskom obliku.

uputa:



$z_1=i-1=-1+i$

$\theta_1=\frac{3\pi}{4}$

$z_1=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

$z=\frac{\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{2\pi}{4}\right)}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}=\ \ldots$

$=\sqrt{2}\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$

Matrice

Neka su m i n pozitivni cijeli brojevi.
$m \times n$ matrica je kolekcija od $m \cdot n$ brojeva uređenih u pravougaoni niz:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \text{ m redova}$$

n kolona

Npr. $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -5 \end{bmatrix}$ je 2×3 matrica, $A = \begin{bmatrix} 1 & \sqrt{2} & 8 & 9 \\ 7 & 2 & -5 & 3 \\ 4 & -6 & 7 & 8 \\ 3 & 7 & 2 & 8 \\ 1 & 2 & -2 & 5 \end{bmatrix}_{5 \times 4}$

Brojeve u matrici zovemo elementi matrice i označavamo sa a_{ij}, gdje su i, j cijeli $1 \leq i \leq m$ i $1 \leq j \leq n$. Indeks i zovemo red indeks, a j kolona indeks.

$$\begin{array}{cc} & j \\ i & \begin{bmatrix} & \vdots & \\ \dots & a_{ij} & \dots\dots \\ & \vdots & \end{bmatrix} \end{array}$$

Npr. u matrici A

$a_{12} = \sqrt{2}$, $a_{23} = -5$, $a_{43} = 2$, $a_{53} = -2$

$1 \times n$ matricu zovemo n-dimenzionalni red vektor, $A = [a_1 \dots a_n]$
$m \times 1$ matricu je m-dimenzionalni kolona vektor

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Sabiranje matrica: $[a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [s_{ij}]_{m \times n}$,

gdje je $s_{ij} = a_{ij} + b_{ij}$, $\forall i, j$

npr.

$$\begin{bmatrix} 2 & 1 & 0 & 3 \\ 4 & 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 4 & 1 & 3 & 4 \end{bmatrix}$$

Skalarno množenje matrice brojem:

c je realan broj $\quad c\cdot[a_{ij}]_{m\times n} = [b_{ij}]_{m\times n}$

npr.

$$2\begin{bmatrix}0&1\\2&3\\2&1\end{bmatrix} = \begin{bmatrix}0&2\\4&6\\4&2\end{bmatrix}$$

gdje je $b_{ij} = c\cdot a_{ij}\ \forall_{i,j}$

Brojeve ćemo često zvati skalari.

Množenje matrica:

Prvo ćemo vidjeti šta je proizvod red vektora A i kolone vektora B.

$$A\cdot B = [a_1\ a_2 \dots a_n]\begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix} = a_1b_1 + a_2b_2 + \dots + a_mb_m.$$

npr. $[3\ \ 1\ \ 2]\begin{bmatrix}1\\-1\\4\end{bmatrix} = 3-1+8=10$

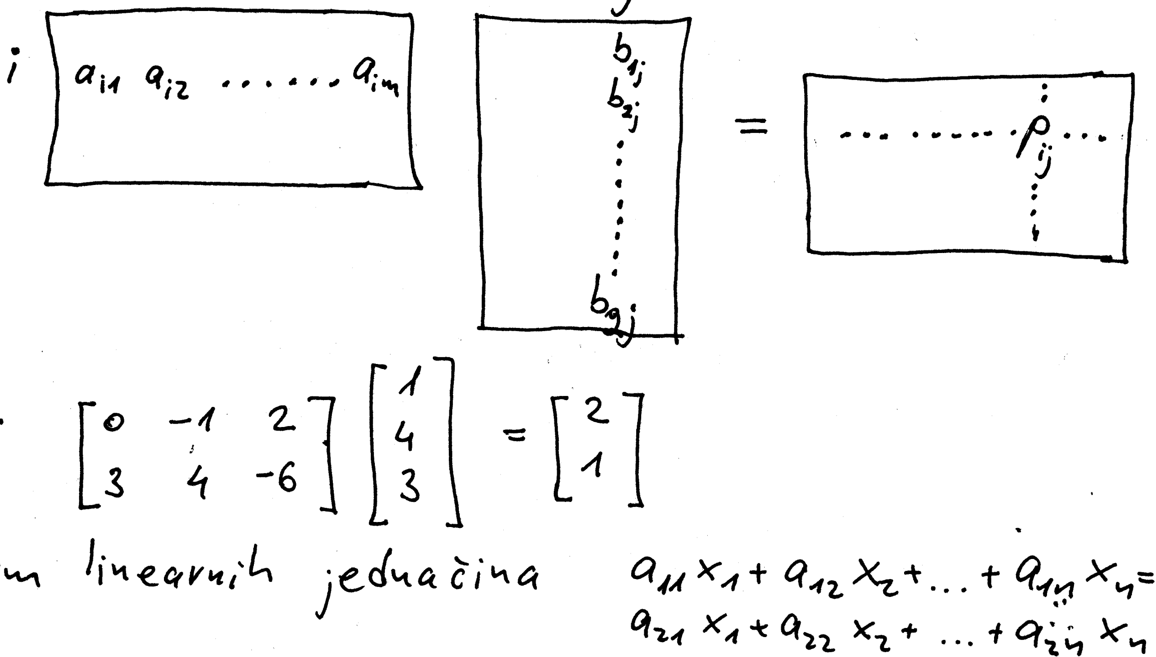
generalno:

$$[a_{ij}]_{m\times q}\cdot[b_{ij}]_{q\times s} = [p_{ij}]_{m\times s}$$

gdje je

$$p_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{im}b_{mj}$$

ovo znači proizvod i-tog reda od A i j-te kolone od B.



npr. $\begin{bmatrix}0&-1&2\\3&4&-6\end{bmatrix}\begin{bmatrix}1\\4\\3\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$

Sistem linearnih jednačina

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1\\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2\\ \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad &\quad\ \vdots\\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m\end{aligned}$$

možemo pisati u matričnom obliku $Ax = \mathbf{b}$, gdje A predstavlja koeficijent matricu $[a_{ij}]_{m\times n}$

$$\boxed{\quad A \quad}\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} = \begin{bmatrix}b_1\\\vdots\\b_m\end{bmatrix}.$$

① Ako je $A=\begin{bmatrix}2&4&5\\3&2&6\\1&1&7\end{bmatrix}$ i $B=\begin{bmatrix}1&-1&6\\3&0&4\\5&2&10\end{bmatrix}$ izračunati:

a) $A+B$ b) $A-B$ c) $2A-3B-I$ (I jedinična matrica)

Rj. a) $\begin{bmatrix}2&4&5\\3&2&6\\1&1&7\end{bmatrix}+\begin{bmatrix}1&-1&6\\3&0&4\\5&2&10\end{bmatrix}=\begin{bmatrix}3&3&11\\6&2&10\\6&3&17\end{bmatrix}$ b) $\begin{bmatrix}2&4&5\\3&2&6\\1&1&7\end{bmatrix}-\begin{bmatrix}1&-1&6\\3&0&4\\5&2&10\end{bmatrix}=\begin{bmatrix}1&5&-1\\0&2&2\\-4&-1&-3\end{bmatrix}$

c) $$2\begin{bmatrix}2&4&5\\3&2&6\\1&1&7\end{bmatrix}-3\begin{bmatrix}1&-1&6\\3&0&4\\5&2&10\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}4&8&10\\6&4&12\\2&2&14\end{bmatrix}-\begin{bmatrix}3&-3&18\\9&0&12\\15&6&30\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&11&-8\\-3&3&0\\-13&-4&-17\end{bmatrix}$$

② Izračunati:

a) $$\begin{bmatrix}2&3\\1&6\\0&1\end{bmatrix}\cdot\begin{bmatrix}2&1\\3&5\end{bmatrix}$$ Rj. $$=\begin{bmatrix}2\cdot2+3\cdot3&2\cdot1+3\cdot5\\1\cdot2+6\cdot3&1\cdot1+6\cdot5\\0\cdot2+1\cdot3&0\cdot1+1\cdot5\end{bmatrix}=\begin{bmatrix}13&17\\20&31\\3&5\end{bmatrix}$$

b) $$\begin{bmatrix}1&4\\2&-5\\3&6\end{bmatrix}\begin{bmatrix}1&4&-2\\2&5&6\end{bmatrix}=\begin{bmatrix}1\cdot1+4\cdot2&1\cdot4+4\cdot5&1\cdot(-2)+4\cdot6\\2\cdot1+(-5)\cdot2&2\cdot4+(-5)\cdot5&2\cdot(-2)+(-5)6\\3\cdot1+6\cdot2&3\cdot4+6\cdot5&3\cdot(-2)+6\cdot6\end{bmatrix}=\begin{bmatrix}9&24&22\\-8&-17&-34\\15&42&30\end{bmatrix}$$

c) $$\begin{bmatrix}2\\1\\3\end{bmatrix}\begin{bmatrix}1&2&3\end{bmatrix}=\begin{bmatrix}2&4&6\\1&2&3\\3&6&9\end{bmatrix}$$ d) $$\begin{bmatrix}a&b&c\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}=a+2b+3c$$

③ Ako je $A=\begin{bmatrix}1&-2&3\\2&-4&1\\3&-5&2\end{bmatrix}$ izračunati $3A^2-2A^T+5\cdot I$.

(A^T transponovana matrica matrice A) (kada elementi iz reda zamjene položaj sa elementima iz kolona)

Rj. $$A^T=\begin{bmatrix}1&2&3\\-2&-4&-5\\3&1&2\end{bmatrix}.\quad A^2=\begin{bmatrix}1&-2&3\\2&-4&1\\3&-5&2\end{bmatrix}\begin{bmatrix}1&-2&3\\2&-4&1\\3&-5&2\end{bmatrix}=\begin{bmatrix}6&-9&7\\-3&7&4\\-1&4&8\end{bmatrix}$$

$$3A^2-2A^T+5=\begin{bmatrix}18&-27&21\\-9&21&12\\-3&12&24\end{bmatrix}-\begin{bmatrix}2&4&6\\-4&-8&-10\\6&2&4\end{bmatrix}+\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}=\begin{bmatrix}21&-31&15\\-5&34&22\\-9&10&25\end{bmatrix}$$

④ Ako je $A=\begin{bmatrix}2&3&5\\-3&1&5\end{bmatrix}$ i $B=\begin{bmatrix}-2&-3\\-1&0\\1&1\end{bmatrix}$, izračunati $2\cdot A^T\cdot A-3\cdot B\cdot B^T+6I$.

Rj. $\begin{bmatrix}-7&0&5\\0&23&43\\5&43&100\end{bmatrix}$

Determinante

matrica tipa n×n

Determinanta je broj pridružen svakoj kvadratnoj matrici.
Determinantu matrice A obilježavamo sa $\det A$ ili $|A|$.
Preciznija definicija determinante je:
Determinanta je f-ja koja n×n realnih brojeva preslikava u realan broj.

Osobine determinante: (neke osobine determinanti)

1. Determinanta jedinične matrice je 1 ($\det I = 1$).
2. Ako dva reda (ili dvije kolone) međusobno zamjene mjesta znak determinante se mijenja.

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \quad \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3. a) Determinanta se množi jednim brojem ako se tim brojem pomnože svi elementi jednog reda (ili jedne kolone)

$$t\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} ta & tb \\ c & d \end{vmatrix}$$

b) $$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

(linearnost za svaki red)

① Izračunati:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

a) $$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{vmatrix} \underset{\downarrow}{\overset{r_3}{=}} 2\cdot\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 0\cdot\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 0\cdot\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 2\cdot 1 = 2$$

razvoj determinante po trećem redu

b) $$\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} \underset{\text{po prvom redu}}{\overset{\text{razvoj determinante}}{=}} 1\cdot\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 2\cdot\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} + 0\cdot\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}$$

$$= 1\cdot 0 - 2\cdot(-3) + 0 = 6$$

Mogli smo izračunati i na sljedeći način:

$$\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} \xlongequal{II_k - III_k} \begin{vmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 1 \end{vmatrix} = (-2)\cdot\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = (-2)\cdot(-3) = 6$$

② Izračunati:

a) $$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} \xlongequal{III_k - IV_k} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2$$

b) $$\begin{vmatrix} 4 & 1 & 0 & 3 \\ 2 & 0 & 1 & 1 \\ 4 & 1 & 0 & 1 \\ 0 & 1 & 0 & 3 \end{vmatrix} \xlongequal{I_R - IV_R} \begin{vmatrix} 4 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 4 & 1 & 0 & 1 \\ 0 & 1 & 0 & 3 \end{vmatrix} = 4 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 3 \end{vmatrix} = 4 \cdot (-1) \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = (-4) \cdot 2 =$$
$$= -8$$

③ Izračunati:

a) $$\begin{vmatrix} 3 & -2 & 1 \\ 4 & -1 & 1 \\ 1 & 1 & 5 \end{vmatrix} \xlongequal[\text{Rj:}]{I_R - II_R} \begin{vmatrix} -1 & -1 & 0 \\ 4 & -1 & 1 \\ 1 & 1 & 5 \end{vmatrix} \xlongequal{III_R + I_R} \begin{vmatrix} -1 & -1 & 0 \\ 4 & -1 & 1 \\ 0 & 0 & 5 \end{vmatrix} = 5 \begin{vmatrix} -1 & -1 \\ 4 & -1 \end{vmatrix}$$
$$= 5 \cdot 5 = 25$$

b) $$\begin{vmatrix} 1 & 3 & 3 \\ 2 & -1 & 4 \\ 1 & 2 & 7 \end{vmatrix} \xlongequal{I_R - III_R} \begin{vmatrix} 0 & 1 & -4 \\ 2 & -1 & 4 \\ 1 & 2 & 7 \end{vmatrix} \xlongequal{II_R + I_R} \begin{vmatrix} 0 & -1 & -4 \\ 2 & 0 & 0 \\ 1 & 2 & 7 \end{vmatrix} = (-2) \cdot \begin{vmatrix} 1 & -4 \\ 2 & 7 \end{vmatrix}$$
$$= (-2) \cdot 15 = -30$$

④ Izračunati:

a) Rj:
$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 2 & 5 & 2 & 0 \\ 3 & 0 & 0 & 1 \\ 4 & 3 & 2 & 1 \end{vmatrix} \xlongequal[\substack{III_R - I_R \cdot 3 \\ IV_R - I_R \cdot 4}]{II_R - I_R \cdot 2} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 3 & -2 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 5 & 0 & 0 \\ 0 & -3 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 5 \cdot \begin{vmatrix} -3 & 1 \\ -2 & 1 \end{vmatrix}$$
$$= 5 \cdot (-1) = -5$$

b)^{v} $$\begin{vmatrix} 0 & 0 & 1 & 2 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 \end{vmatrix}$$

c)^{v} $$\begin{vmatrix} 5 & 4 & 3 & 2 \\ 1 & 1 & 2 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \end{vmatrix}$$

Rješenja:
b) 0 c) -1

⑤^{v} Izračunati: $$\begin{vmatrix} \sqrt{3} & 2\sqrt{2} & \sqrt{5} \\ 5\sqrt{3} & \sqrt{8} & 7\sqrt{5} \\ \sqrt{5}+2\sqrt{3} & 4\sqrt{2} & \sqrt{3}+2\sqrt{5} \end{vmatrix}.$$

Rj. $36\sqrt{2}$

⑥ Dokazati da je $\begin{vmatrix} 1 & a & a^2+a^3 \\ 1 & a^2 & a^3+a \\ 1 & a^3 & a+a^2 \end{vmatrix} = 0$.

Rj.

$$\begin{vmatrix} 1 & a & a^2+a^3 \\ 1 & a^2 & a^3+a \\ 1 & a^3 & a+a^2 \end{vmatrix} = a \begin{vmatrix} 1 & 1 & a^2(1+a) \\ 1 & a & a(a^2+1) \\ 1 & a^2 & a(1+a) \end{vmatrix} = a \cdot a \cdot \begin{vmatrix} 1 & 1 & a(a+1) \\ 1 & a & a^2+1 \\ 1 & a^2 & a+1 \end{vmatrix} \overset{II_R - I_R}{\underset{III_R - I_R}{=\!=}}$$

$$= a^2 \begin{vmatrix} 1 & 1 & a(a+1) \\ 0 & a-1 & 1-a \\ 0 & a^2-1 & 1-a^2 \end{vmatrix} = a^2 \begin{vmatrix} a-1 & 1-a \\ (a+1)(a-1) & 1-a^2 \end{vmatrix} = a^2(a-1) \begin{vmatrix} 1 & 1-a \\ a+1 & (1-a)(1+a) \end{vmatrix}$$

$$= a^2(a-1)(1-a) \begin{vmatrix} 1 & 1 \\ a+1 & a+1 \end{vmatrix} = a^2(a-1)(1-a)(a+1) \underbrace{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}_{=0} = 0$$

što je i trebalo dobiti.

⑦ Izračunati: $\begin{vmatrix} a & b & a & b \\ b & a & a & b \\ a & b & b & a \\ b & a & b & a \end{vmatrix}$

Rj. $\overset{IV_k + (I_k + II_k + III_k)}{=\!=\!=} \begin{vmatrix} a & b & a & 2a+2b \\ b & a & a & 2a+2b \\ a & b & b & 2a+2b \\ b & a & b & 2a+2b \end{vmatrix}$

$$= (2a+2b) \begin{vmatrix} a & b & a & 1 \\ b & a & a & 1 \\ a & b & b & 1 \\ b & a & b & 1 \end{vmatrix} \overset{II_R - I_R}{\underset{IV_R - III_R}{=\!=}} (2a+2b) \begin{vmatrix} a & b & a & 1 \\ b-a & a-b & 0 & 0 \\ a & b & b & 1 \\ b-a & a-b & 0 & 0 \end{vmatrix} \overset{III_R - I_R}{=\!=} (2a+2b) \cdot$$

$$\cdot \begin{vmatrix} a & b & a & 1 \\ b-a & a-b & 0 & 0 \\ 0 & 0 & b-a & 1 \\ b-a & a-b & 0 & 0 \end{vmatrix} = -(2a+2b) \cdot \begin{vmatrix} a & b & a \\ b-a & a-b & 0 \\ b-a & a-b & 0 \end{vmatrix} = -a(2a+2b) \begin{vmatrix} b-a & a-b \\ b-a & a-b \end{vmatrix}$$

$$= -a(2a+2b)(b-a)(a-b) \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0.$$

⑧ Rastaviti na faktore polinom:

a) $\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$ b) $\begin{vmatrix} a & b & a+b \\ b & a+b & a \\ a+b & a & b \end{vmatrix}$ c) $\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 \\ b^2 & (b+1)^2 & (b+2)^2 \\ c^2 & (c+1)^2 & (c+2)^2 \end{vmatrix}$.

(#) Riješiti jednačinu $\begin{vmatrix}3x-5 & -5-2x & x+1\\ 2x-4 & -2-2x & x-1\\ 3x-8 & 2-3x & 2x-5\end{vmatrix}=0$.

Rj.
$$\begin{vmatrix}3x-5 & -5-2x & x+1\\ 2x-4 & -2-2x & x-1\\ 3x-8 & 2-3x & 2x-5\end{vmatrix}=(-1)\begin{vmatrix}3x-5 & 2x+5 & x+1\\ 2x-4 & 2x+2 & x-1\\ 3x-8 & 3x-2 & 2x-5\end{vmatrix}\xlongequal{III_v-II_v}$$

$$\begin{vmatrix}3x-5 & 2x+5 & x+1\\ 2x-4 & 2x+2 & x-1\\ x-4 & x-4 & x-4\end{vmatrix}=(-1)(x-4)\begin{vmatrix}3x-5 & 2x+5 & x+1\\ 2x-4 & 2x+2 & x-1\\ 1 & 1 & 1\end{vmatrix}\underset{II_k-III_k}{\overset{I_k-III_k}{=\!=\!=}}$$

$$=(-1)(x-4)\begin{vmatrix}2x-6 & x+4 & x+1\\ x-3 & x+3 & x-1\\ 0 & 0 & 1\end{vmatrix}=(-1)(x-4)\begin{vmatrix}2x-6 & x+4\\ x-3 & x+3\end{vmatrix}\xlongequal{I_v-II_v}$$

$$=(-1)(x-4)\begin{vmatrix}x-3 & 1\\ x-3 & x+3\end{vmatrix}=(-1)(x-4)(x-3)\begin{vmatrix}1 & 1\\ 1 & x+3\end{vmatrix}=(-1)(x-4)(x-3)(x+2)$$

$(-1)(x-4)(x-3)(x+2)=0$ Rješenja jednačine su
$x=4$ ili $x=3$ ili $x=-2$.

(#) Riješiti jednačinu: $\begin{vmatrix}x-3 & x+2 & x-1\\ x+2 & x-4 & x\\ x-1 & x+4 & x-5\end{vmatrix}=0$.

Rj.
$$\begin{vmatrix}x-3 & x+2 & x-1\\ x+2 & x-4 & x\\ x-1 & x+4 & x-5\end{vmatrix}\xlongequal{I_R+II_R+III_R}\begin{vmatrix}3x-2 & x+2 & x-1\\ 3x-2 & x-4 & x\\ 3x-2 & x+4 & x-5\end{vmatrix}=(3x-2)\begin{vmatrix}1 & x+2 & x-1\\ 1 & x-4 & x\\ 1 & x+4 & x-5\end{vmatrix}$$

$$\underset{III_R-II_R}{\overset{I_R-II_R}{=\!=\!=}}(3x-2)\begin{vmatrix}0 & 6 & -1\\ 1 & x-4 & x\\ 0 & 8 & -5\end{vmatrix}=-(3x-2)\begin{vmatrix}6 & -1\\ 8 & -5\end{vmatrix}=-(3x-2)(-30+8)=$$

$=22(3x-2)$

$22(3x-2)=0$

$3x-2=0$

$3x=2$

$x=\frac{2}{3}$ je rješenje jednačine

(#) Izračunati $\begin{vmatrix} 1 & a & 3 & 2 \\ 2 & 2 & -2 & 1 \\ 3 & 3 & -5 & 1 \\ 4 & 4 & -7 & 5 \end{vmatrix}$.

Rj.

$$\begin{vmatrix} 1 & a & 3 & 2 \\ 2 & 2 & -2 & 1 \\ 3 & 3 & -5 & 1 \\ 4 & 4 & -7 & 5 \end{vmatrix} \xlongequal[III_k+IV_k\cdot 2]{\substack{I_k+II_k \\ II_k+III_k}} \begin{vmatrix} 4 & a+3 & 7 & 2 \\ 0 & 0 & 0 & \boxed{1} \\ -2 & -2 & -3 & 1 \\ -3 & -3 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 4 & a+3 & 7 \\ -2 & -2 & -3 \\ -3 & -3 & 3 \end{vmatrix} \xlongequal[II_k+III_k]{I_k+III_k}$$

$$= \begin{vmatrix} 11 & a+10 & 7 \\ -5 & -5 & -3 \\ 0 & 0 & 3 \end{vmatrix} = 3 \begin{vmatrix} 11 & a+10 \\ -5 & -5 \end{vmatrix} = 3\cdot(-5) \begin{vmatrix} 11 & a+10 \\ 1 & 1 \end{vmatrix} = -15\,(11-a-10)$$

$$= -15\,(-a+1) = 15a - 15$$

(#) Matematičkom indukcijom dokazati:

$$\begin{vmatrix} 1+x^2 & x & 0 & \cdots & 0 & 0 \\ x & 1+x^2 & x & \cdots & 0 & 0 \\ 0 & x & 1+x^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+x^2 & x \\ 0 & 0 & 0 & \cdots & x & 1+x^2 \end{vmatrix} = 1+x^2+x^4+\ldots+x^{2n}$$

(determinanta ima n vrsta i n kolona).

R: BAZA INDUKCIJE

Pokažimo da je tvrdnja tačna za broj 2

$$\begin{vmatrix} 1+x^2 & x \\ x & 1+x^2 \end{vmatrix} = (1+x^2)^2 - x^2 = 1+2x^2+x^4-x^2 = 1+x^2+x^4$$

Jednakost je tačna za broj 2.

KORAK INDUKCIJE

Pretpostavimo da je jednakost tačna za determinantu koja ima k vrsta i k kolona

$$\begin{vmatrix} 1+x^2 & x & \cdots & 0 & 0 \\ x & 1+x^2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1+x^2 & x \\ 0 & 0 & \cdots & x & 1+x^2 \end{vmatrix} = 1+x^2+x^4+\ldots+x^{2k}$$

gdje k uzima brojeve od 1 do n.

Na osnovu ove pretpostavke dokažimo da je jednakost tačna za determinantu koja ima n+1 vrstu i n+1 kolonu tačnije dokažimo da

$$\begin{vmatrix} 1+x^2 & x & \cdots & 0 & 0 \\ x & 1+x^2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1+x^2 & x \\ 0 & 0 & \cdots & x & 1+x^2 \end{vmatrix} = 1+x^2+x^4+\ldots+x^{2n}+x^{2n+2}.$$

Polazimo od determinante koja ima (n+1)-vrstu i (n+1)-kolonu:

$$\begin{vmatrix} 1+x^2 & x & \cdots & 0 & 0 \\ x & 1+x^2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1+x^2 & x \\ 0 & 0 & \cdots & x & 1+x^2 \end{vmatrix} \overset{\text{razvoj po prvoj}}{\underset{\text{koloni}}{=}} (1+x^2)\begin{vmatrix} 1+x^2 & x & \cdots & 0 & 0 \\ x & 1+x^2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1+x^2 & x \\ 0 & 0 & \cdots & x & 1+x^2 \end{vmatrix} - x\begin{vmatrix} x & 0 & 0 & \cdots & 0 & 0 \\ x & 1+x^2 & x & \cdots & 0 & 0 \\ 0 & x & 1+x^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+x^2 & x \\ 0 & 0 & 0 & \cdots & x & 1+x^2 \end{vmatrix} =$$

(ova determinanta ima n vrsta i n kolona)

(ovu determinantu mogu razviti po prvoj vrsti i ostaće mi determinanta iz pretpostavke koja ima n-1 vrstu i n-1 kolonu)

$$\overset{\text{na osnovu}}{\underset{\text{pretpostavke}}{=}} (1+x^2)(1+x^2+x^4+\ldots+x^{2n}) - x^2(1+x^2+x^4+\ldots+x^{2n-2})$$

$$= (1+x^2+x^4+\ldots+x^{2n}) + (x^2+x^4+x^6+\ldots+x^{2n}+x^{2n+2}) -$$

$$-(x^2+x^4+x^6+\ldots+x^{2n-2}+x^{2n}) = 1+x^2+x^4+\ldots+x^{2n+2}$$ što je i trebalo dobiti

ZAKLJUČAK

Jednakost je tačna za sve prirodne brojeve

#) Matematičkom indukcijom dokazati:

$$\begin{vmatrix} 1 & n & n & \dots & n & n \\ n & 2 & n & \dots & n & n \\ n & n & 3 & \dots & n & n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & n & n & \dots & n-1 & n \\ n & n & n & \dots & n & n \end{vmatrix} = (-1)^{n-1} \cdot n!$$

Rj. BAZA INDUKCIJE

Pokažimo da je tvrdnja tačna za broj 2.

$$\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 2-4 = -2 = (-1)^{2-1} \cdot 2!$$ Jednakost je tačna za broj 2.

KORAK INDUKCIJE

Pretpostavimo da je jednakost

$$\begin{vmatrix} 1 & k & k & \dots & k & k \\ k & 2 & k & \dots & k & k \\ k & k & 3 & \dots & k & k \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ k & k & k & \dots & k-1 & k \\ k & k & k & \dots & k & k \end{vmatrix} = (-1)^{k-1} \cdot k!$$

tačna za sve brojeve od 1 do n $(k=1,2,\dots,n)$.

Uz pomoć ove pretpostavke dokažimo da je jednakost tačna za broj n+1 tj. dokažimo

$$\begin{vmatrix} 1 & n+1 & \dots & n+1 & n+1 \\ n+1 & 2 & \dots & n+1 & n+1 \\ \vdots & \vdots & & \vdots & \vdots \\ n+1 & n+1 & \dots & n & n+1 \\ n+1 & n+1 & \dots & n+1 & n+1 \end{vmatrix} = (-1)^{n} \cdot (n+1)!$$

ZAKLJUČAK

Jednakost je tačna za sve prirodne brojeve

$$\begin{vmatrix} 1 & n+1 & \dots & n+1 & n+1 \\ n+1 & 2 & \dots & n+1 & n+1 \\ \vdots & \vdots & & \vdots & \vdots \\ n+1 & n+1 & \dots & n & n+1 \\ n+1 & n+1 & \dots & n+1 & n+1 \end{vmatrix} \underset{\overline{\overline{}}}{\overset{I_k-(N+1)_k}{=}} \begin{vmatrix} -n & n+1 & \dots & n+1 & n+1 \\ 0 & 2 & \dots & n+1 & n+1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & n+1 & \dots & n & n+1 \\ 0 & n+1 & \dots & n+1 & n+1 \end{vmatrix} =$$

$$= (-n) \begin{vmatrix} 2 & n+1 & \dots & n+1 & n+1 \\ n+1 & 3 & \dots & n+1 & n+1 \\ \vdots & \vdots & & \vdots & \vdots \\ n+1 & n+1 & \dots & n & n+1 \\ n+1 & n+1 & \dots & n+1 & n+1 \end{vmatrix} = (-n)(n+1) \begin{vmatrix} 2 & n+1 & \dots & n+1 & 1 \\ n+1 & 3 & \dots & n+1 & 1 \\ \vdots & \vdots & & \vdots & \\ n+1 & n+1 & \dots & n & 1 \\ n+1 & n+1 & \dots & n+1 & 1 \end{vmatrix} \overset{\begin{matrix} I_k - N_k \\ II_k - N_k \end{matrix}}{=} \begin{matrix} \vdots \\ (N-1)_k - N_k \end{matrix}$$

$$= (-1) \cdot n(n+1) \begin{vmatrix} 1 & n & \dots & n & 1 \\ n & 2 & \dots & n & 1 \\ \vdots & \vdots & & & \\ n & n & \dots & n-1 & 1 \\ n & n & \dots & n & 1 \end{vmatrix} = (-1)(n+1) \begin{vmatrix} 1 & n & \dots & n & n \\ n & 2 & \dots & n & n \\ \vdots & \vdots & & \vdots & \vdots \\ n & n & \dots & n-1 & n \\ n & n & \dots & n & n \end{vmatrix} \overset{\text{na osnovu pretpostavke}}{=} (-1)(n+1)(-1)^{n-1} \cdot n!$$

$$= (-1)^{n}(n+1)!$$

Rang matrice

Minor reda k matrice A je determinanta reda k sastavljena od elemenata koji stoje na presjecima proizvoljnih k vrsta i k kolona matrice A.

Npr.

$$A=\begin{bmatrix}3&4&5&6&7\\1&2&3&4&5\\3&4&7&5&2\\2&3&1&7&5\end{bmatrix}$$

minor reda 3

$$\begin{vmatrix}4&5&6\\2&3&4\\4&7&5\end{vmatrix}$$

minor reda 4

$$\begin{vmatrix}4&5&6&7\\2&3&4&5\\4&7&5&2\\3&1&7&5\end{vmatrix}$$

Rang matrice A je broj (označavamo ga sa rang(A)) koji je jednak redu maksimalnog minora, različitog od nule, determinante det A.

Za dvije matrice A i B kažemo da su ekvivalentne ako imaju isti rang. Rang matrice tražimo elementarnim transformacijama:

1. razmjena mjesta dvije vrste (red) ili dvije kolone
2. dodavanjem elemenata jednog reda (ili jedne kolone) odgovarajućim elementima drugog reda (ili druge kolone) pomnožena nekim brojem.
3. množenje elemenata jednog reda (ili jedne kolone) nekim brojem različitim od nule

Ekvivalentne matrice označavamo sa $A \sim B$.

① Odrediti rang matrice:

a)

$$M=\begin{bmatrix}2&-1&3&-2&4\\4&-2&4&1&7\\2&-1&1&8&2\end{bmatrix}\underset{III_R-I_R}{\overset{\substack{Rj.\\ II_R-I_R\cdot 2}}{\sim}}\begin{bmatrix}2&-1&3&-2&4\\0&0&-2&5&-1\\0&0&-2&10&-2\end{bmatrix}\overset{III_R-II_R}{\sim}\begin{bmatrix}2&-1&3&-2&4\\0&0&-2&5&-1\\0&0&0&5&-1\end{bmatrix},\quad rang(M)=3$$

b)

$$A=\begin{bmatrix}-2&1&0&2\\0&-1&1&3\\-1&1&0&-2\\-4&2&1&1\end{bmatrix}\overset{\substack{Rj.\\ I_K\leftrightarrow II_K}}{\sim}\begin{bmatrix}1&-2&0&2\\-1&0&1&3\\1&-1&0&-2\\2&-4&1&1\end{bmatrix}\underset{\substack{III_R-I_R\\ IV_R-I_R\cdot 2}}{\overset{II_R+I_R}{\sim}}\begin{bmatrix}1&-2&0&2\\0&-2&1&5\\0&1&0&-4\\0&0&1&-3\end{bmatrix}\overset{II_K\leftrightarrow III_K}{\sim}$$

$$\sim\begin{bmatrix}1&0&-2&2\\0&1&-2&5\\0&0&1&-4\\0&1&0&-3\end{bmatrix}\overset{IV_R-I_R}{\sim}\begin{bmatrix}1&0&-2&2\\0&1&-2&5\\0&0&1&-4\\0&0&2&-8\end{bmatrix}\overset{IV_R-III_R\cdot 2}{\sim}\begin{bmatrix}1&0&-2&2\\0&1&-2&5\\0&0&1&-4\\0&0&0&0\end{bmatrix},\quad rang(A)=2$$

② Odrediti rang matrice $A=\begin{bmatrix}1&1&1&1\\2&3&-1&1\\3&4&0&\lambda+2\end{bmatrix}$, $\lambda\in\mathbb{R}$.

Rj.

$$\begin{bmatrix}1&1&1&1\\2&3&-1&1\\3&4&0&\lambda+2\end{bmatrix}\overset{III_k\leftrightarrow I_k}{\sim}\begin{bmatrix}1&1&1&1\\-1&3&2&1\\0&4&3&\lambda+2\end{bmatrix}\overset{II_v+I_v}{\sim}\begin{bmatrix}1&1&1&1\\0&4&3&2\\0&4&3&\lambda+2\end{bmatrix}\overset{III_v-II_v}{\sim}$$

$$\sim\begin{bmatrix}1&1&1&1\\0&4&3&2\\0&0&0&\lambda\end{bmatrix},$$

ako je $\lambda=0$ tada je $rang(A)=2$

ako je $\lambda\neq0$ tada je $rang(A)=3$

③ U ovisnosti o parametru $\lambda\in\mathbb{R}$ odredite rang matrice

$$A=\begin{bmatrix}1&1&1\\1&\lambda&\lambda^2\\1&\lambda^2&\lambda\end{bmatrix}.$$

Rj.

$$\begin{bmatrix}1&1&1\\1&\lambda&\lambda^2\\1&\lambda^2&\lambda\end{bmatrix}\underset{III_R-I_R}{\overset{II_R-I_R}{\sim}}\begin{bmatrix}1&1&1\\0&\lambda-1&\lambda^2-1\\0&\lambda^2-1&\lambda-1\end{bmatrix}\underset{\substack{III_R:(\lambda-1)\\ \lambda\neq1\\ \text{zapamtimo}}}{\overset{II_R:(\lambda-1)}{\sim}}\begin{bmatrix}1&1&1\\0&1&\lambda+1\\0&\lambda+1&1\end{bmatrix}\overset{III_R+II_R\cdot(-\lambda-1)}{\sim}\begin{bmatrix}0&1&1\\0&1&\lambda+1\\0&0&\underbrace{-(\lambda+1)^2+1}_{-(\lambda^2+2\lambda+1)+1}\end{bmatrix}\sim\begin{bmatrix}1&1&1\\0&1&\lambda+1\\0&0&-\lambda(\lambda+2\end{bmatrix}$$

Matrica se ne može više pojednostaviti. Diskusija:

Za $\lambda=0$ dobićemo $\begin{bmatrix}1&1&1\\0&1&1\\0&0&0\end{bmatrix}\Rightarrow rang\,A=2$

Za $\lambda=-2$ imamo $\begin{bmatrix}1&1&1\\0&1&-1\\0&0&0\end{bmatrix}\Rightarrow rang\,A=2$.

Ostaje nam još slučaj $\lambda=1$. Zašto?

Za $\lambda=1$, $A=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\Rightarrow rang\,A=1$. Zašto?

U ostalim slučajevima (tj. kad je $\lambda\neq0$, $\lambda\neq-2$ i $\lambda\neq1$) $rang\,A=3$.

④ Diskutovati rang matrice $M=\begin{bmatrix}1&10&-6&\lambda\\2&-1&\lambda&3\\1&\lambda&-1&2\end{bmatrix}$.

⑤ Diskutovati o rangu matrice

$$M=\begin{bmatrix}a&b&1\\1&ab&1\\1&b&a\end{bmatrix}$$ u zavisnosti od parametara a i b.

(#) Diskutovati rang matrice u zavisnosti od parametara a i b,

$$A = \begin{bmatrix} 2 & 3 & 9 & 6 & 2 \\ 5 & 4 & 12 & 8 & 5 \\ 1 & 2 & 6 & 4 & 1 \\ 4 & 1 & 3 & 2 & a \\ 3 & b & 6 & 4 & 3 \\ 7 & 5 & 15 & 10 & 7 \end{bmatrix}$$

Rj.

$$A = \begin{bmatrix} 2 & 3 & 9 & 6 & 2 \\ 5 & 4 & 12 & 8 & 5 \\ 1 & 2 & 6 & 4 & 1 \\ 4 & 1 & 3 & 2 & a \\ 3 & b & 6 & 4 & 3 \\ 7 & 5 & 15 & 10 & 7 \end{bmatrix} \overset{II_k \leftrightarrow IV_k}{\sim} \begin{bmatrix} 2 & 6 & 9 & 3 & 2 \\ 5 & 8 & 12 & 4 & 5 \\ 1 & 4 & 6 & 2 & 1 \\ 4 & 2 & 3 & 1 & a \\ 3 & 4 & 6 & b & 3 \\ 7 & 10 & 15 & 5 & 7 \end{bmatrix} \overset{IV_R \leftrightarrow VI_R}{\sim} \begin{bmatrix} 2 & 6 & 9 & 3 & 2 \\ 5 & 8 & 12 & 4 & 5 \\ 1 & 4 & 6 & 2 & 1 \\ 7 & 10 & 15 & 5 & 7 \\ 3 & 4 & 6 & b & 3 \\ 4 & 2 & 3 & 1 & a \end{bmatrix} \overset{III_R \leftrightarrow I_R}{\sim}$$

$$\sim \begin{bmatrix} 1 & 4 & 6 & 2 & 1 \\ 5 & 8 & 12 & 4 & 5 \\ 2 & 6 & 9 & 3 & 2 \\ 7 & 10 & 15 & 5 & 7 \\ 3 & 4 & 6 & b & 3 \\ 4 & 2 & 3 & 1 & a \end{bmatrix} \overset{\substack{II_R - I_R \cdot 5 \\ III_R - I_R \cdot 2}}{\underset{\substack{IV_R - I_R \cdot 7 \\ V_R - I_R \cdot 3 \\ VI_R - I_R \cdot 4}}{\sim}} \begin{bmatrix} 1 & 4 & 6 & 2 & 1 \\ 0 & -12 & -18 & -6 & 0 \\ 0 & -2 & -3 & -1 & 0 \\ 0 & -18 & -27 & -9 & 0 \\ 0 & -8 & -12 & b-6 & 0 \\ 0 & -14 & -21 & -7 & a-4 \end{bmatrix} \overset{II_k \leftrightarrow V_k}{\sim} \begin{bmatrix} 1 & 1 & 6 & 2 & 4 \\ 0 & 0 & -18 & -6 & -12 \\ 0 & 0 & -3 & -1 & -2 \\ 0 & 0 & -27 & -9 & -18 \\ 0 & 0 & -12 & b-6 & -8 \\ 0 & a-4 & -21 & -7 & -14 \end{bmatrix} \sim$$

$$\overset{II_R \leftrightarrow III_R}{\sim} \begin{bmatrix} 1 & 1 & 6 & 2 & 4 \\ 0 & 0 & -3 & -1 & -2 \\ 0 & 0 & -18 & -6 & -12 \\ 0 & 0 & -27 & -9 & -18 \\ 0 & 0 & -12 & b-6 & -8 \\ 0 & a-4 & -21 & -7 & -14 \end{bmatrix} \overset{III_R - II_R \cdot 6}{\underset{\substack{IV_R - II_R \cdot 9 \\ V_R - II_R \cdot 4 \\ VI_R - II_R \cdot 7}}{\sim}} \begin{bmatrix} 1 & 1 & 6 & 2 & 4 \\ 0 & 0 & -3 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b-2 & 0 \\ 0 & a-4 & 0 & 0 & 0 \end{bmatrix}$$

Diskusija

1° $a=4$, $b=2$ rang $A = 2$

2° $a=4$, $b\neq 2$ rang $A = 3$

3° $a\neq 4$, $b=2$ rang $A = 3$

4° $a\neq 4$, $b\neq 2$ rang $A = 4$

(#) Diskutovati rang matrice $M=\begin{bmatrix} 14 & 4 & 2\lambda-4 & -6 \\ 6 & 2 & -1 & -3 \\ 3\lambda+4 & 2 & -2\lambda+1 & -3 \\ 24 & 8 & -4 & -12 \end{bmatrix}$

za razne vrijednosti parametra λ.

Rj.

$$M=\begin{bmatrix} 14 & 4 & 2\lambda-4 & -6 \\ 6 & 2 & -1 & -3 \\ 3\lambda+4 & 2 & -2\lambda+1 & -3 \\ 24 & 8 & -4 & -12 \end{bmatrix} \overset{III_v+I_v}{\sim} \begin{bmatrix} 14 & 4 & 2\lambda-4 & -6 \\ 6 & 2 & -1 & -3 \\ 3\lambda+18 & 6 & -3 & -9 \\ 24 & 8 & -4 & -12 \end{bmatrix} \underset{\substack{I_v:2 \\ III_v:3}}{\overset{IV_v:4}{\sim}}$$

$$\begin{bmatrix} 7 & 2 & \lambda-2 & -3 \\ 6 & 2 & -1 & -3 \\ \lambda+6 & 2 & -1 & -3 \\ 6 & 2 & -1 & -3 \end{bmatrix} \overset{IV_v-II_v}{\sim} \begin{bmatrix} 7 & 2 & \lambda-2 & -3 \\ 6 & 2 & -1 & -3 \\ \lambda+6 & 2 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{I_v\leftrightarrow II_v}{\sim}$$

$$\begin{bmatrix} 6 & 2 & -1 & -3 \\ 7 & 2 & \lambda-2 & -3 \\ \lambda+6 & 2 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{I_k\leftrightarrow IV_k}{\sim} \begin{bmatrix} -3 & 2 & -1 & 6 \\ -3 & 2 & \lambda-2 & 7 \\ -3 & 2 & -1 & \lambda+6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{III_v-I_v}{\overset{II_v-I_v}{\sim}}$$

$$\sim \begin{bmatrix} -3 & 2 & -1 & 6 \\ 0 & 0 & \lambda-1 & 1 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Za $\lambda=0$

$\text{rang}(M)=2$

Za $\lambda\neq 0$ $\text{rang}(M)=3$

(#) Diskutovati rang matrice $\begin{bmatrix} 1 & 2 & t & 0 & -1 \\ 2 & 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 4 & -2 \\ 1 & 0 & 0 & -3 & 4 \end{bmatrix}$ za razne vrijednosti parametra t.

Rj.

$$M = \begin{bmatrix} 1 & 2 & t & 0 & -1 \\ 2 & 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 4 & -2 \\ 1 & 0 & 0 & -3 & 4 \end{bmatrix} \overset{III_k \leftrightarrow V_k}{\sim} \begin{bmatrix} 1 & 2 & -1 & 0 & t \\ 2 & 0 & 2 & 1 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 1 & 0 & 4 & -3 & 0 \end{bmatrix} \overset{I_v \leftrightarrow IV_v}{\sim}$$

$$\begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 2 & 0 & 2 & 1 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 1 & 2 & -1 & 0 & t \end{bmatrix} \underset{IV_v - I_v}{\overset{II_v - I_v \cdot 2}{\sim}} \begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 0 & 0 & -6 & 7 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 0 & 2 & -5 & 3 & t \end{bmatrix} \overset{II_v \leftrightarrow III_v}{\sim} \begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 0 & 0 & -6 & 7 & 0 \\ 0 & 2 & -5 & 3 & t \end{bmatrix}$$

$$\overset{IV_v + II_v \cdot 2}{\sim} \begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 0 & 0 & -6 & 7 & 0 \\ 0 & 0 & -9 & 11 & t \end{bmatrix} \overset{IV_v - III_v \cdot \frac{3}{2}}{\sim} \begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 0 & 0 & -6 & 7 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & t \end{bmatrix}$$

$$-9 + 6 \cdot \frac{3}{2} = -9 + 9 = 0$$

$$11 - 7 \cdot \frac{3}{2} = \frac{22}{2} - \frac{21}{2} = \frac{1}{2}$$

Bez obzira na vrijednost parametra t rang matrice M je uvijek 4.

Inverzna matrica

Transponovanu matricu matrice A obilježavamo sa A^T. Kofaktor A_{ij}, matrice A, elementa a_{ij} je determinanta pomnožena sa $(-1)^{i+j}$ čiji su elementi svi elementi iz matrice A osim one kolone i one vrste u kojoj se nalazi koeficijent a_{ij}.

kofaktor elementa a_{12}

Npr.

$$A=\begin{bmatrix}3&7&2\\6&8&9\\1&2&4\end{bmatrix}, \quad A_{23}=(-1)^{2+3}\begin{vmatrix}3&7\\1&2\end{vmatrix}, \quad A_{12}=(-1)^{1+2}\begin{vmatrix}3&2\\1&4\end{vmatrix}, \quad A_{31}=(-1)^{3+1}\cdot\begin{vmatrix}7&2\\8&9\end{vmatrix}$$

kofaktor elementa a_{23}

kofaktor elementa a_{31}

$$A^T=\begin{bmatrix}3&6&1\\7&8&2\\2&9&4\end{bmatrix}$$

Kofaktor matrica (A_{kof}) kvadratne matrice A je matrica fofaktora A_{ik} elemenata a_{ik} dane matrice.

$$A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix} \qquad A_{kof}=\begin{bmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{bmatrix}$$

Za matricu A kažemo da je regularna ako je $\det A \neq 0$. Inverznu matricu računamo po formuli

$$A^{-1}=\frac{1}{\det A}\cdot A_{kof}^{T}$$

Neke osobine inverzne matrice:

$$A^{-1}\cdot A = A\cdot A^{-1} = I$$

$$(AB)^{-1}=B^{-1}A^{-1}$$

① Naći inverznu matricu matrice $A=\begin{bmatrix}1&0&2\\1&1&0\\1&1&1\end{bmatrix}$.

Rj. $A^{-1}=\dfrac{1}{\det A}\cdot A_{kof}^{T}$

$$\det A=\begin{vmatrix}1&0&2\\1&1&0\\1&1&1\end{vmatrix} \overset{III_R-II_R}{=\!=\!=} \begin{vmatrix}1&0&2\\1&1&0\\0&0&1\end{vmatrix} = 1\cdot\begin{vmatrix}1&0\\1&1\end{vmatrix}=1$$

$A_{11}=(-1)^2\begin{vmatrix}1&0\\1&1\end{vmatrix}=1 \qquad A_{13}=(-1)^4\begin{vmatrix}1&1\\1&1\end{vmatrix}=0 \qquad A_{22}=(-1)^4\begin{vmatrix}1&2\\1&1\end{vmatrix}=-1$

$A_{12}=(-1)^3\begin{vmatrix}1&0\\1&1\end{vmatrix}=-1 \qquad A_{21}=(-1)^3\begin{vmatrix}0&2\\1&1\end{vmatrix}=2 \qquad A_{23}=(-1)^5\begin{vmatrix}1&0\\1&1\end{vmatrix}=-1$

$A_{31}=(-1)^4\begin{vmatrix}0&2\\1&0\end{vmatrix}=-2$

$A_{32}=(-1)^5\begin{vmatrix}1&2\\1&0\end{vmatrix}=2$

$A_{33}=(-1)^6\begin{vmatrix}1&0\\1&1\end{vmatrix}=1$

$$A_{kof}=\begin{bmatrix}1&-1&0\\2&-1&-1\\-2&2&1\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}1&2&-2\\-1&-1&2\\0&-1&1\end{bmatrix}$$

inverzna matrica matrice A

provjera:

$$\begin{bmatrix}1&0&2\\1&1&0\\1&1&1\end{bmatrix}\cdot\begin{bmatrix}1&2&-2\\-1&-1&2\\0&-1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

② Naći inverznu matricu matrice $B=\begin{bmatrix}3&2&4\\2&3&4\\1&1&2\end{bmatrix}$.

Rj. $B^{-1}=\dfrac{1}{\det B}B_{kof}^{T}$, $\det B=\begin{vmatrix}3&2&4\\2&3&4\\1&1&2\end{vmatrix}\dfrac{II_k-I_k}{III_k-I_k\cdot 2}\begin{vmatrix}3&-1&-2\\2&1&0\\1&0&0\end{vmatrix}=\begin{vmatrix}-1&-2\\1&0\end{vmatrix}=2$

$B_{11}=(-1)^2\begin{vmatrix}3&4\\1&2\end{vmatrix}=2$ $\quad B_{21}=(-1)^2\begin{vmatrix}2&4\\1&2\end{vmatrix}=0$ $\quad B_{31}=(-1)^4\begin{vmatrix}2&4\\3&4\end{vmatrix}=-4$

$B_{12}=(-1)^3\begin{vmatrix}2&4\\1&2\end{vmatrix}=0$ $\quad B_{22}=(-1)^4\begin{vmatrix}3&4\\1&2\end{vmatrix}=2$ $\quad B_{32}=(-1)^5\begin{vmatrix}3&4\\2&4\end{vmatrix}=-4$

$B_{13}=(-1)^4\begin{vmatrix}2&3\\1&1\end{vmatrix}=-1$ $\quad B_{23}=(-1)^5\begin{vmatrix}3&2\\1&1\end{vmatrix}=-1$ $\quad B_{33}=(-1)^6\begin{vmatrix}3&2\\2&3\end{vmatrix}=5$

$$B_{kof}^{T}=\begin{bmatrix}2&0&-4\\0&2&-4\\-1&-1&5\end{bmatrix},\quad B^{-1}=\frac{1}{2}\begin{bmatrix}2&0&-4\\0&2&-4\\-1&-1&5\end{bmatrix}=\begin{bmatrix}1&0&-2\\0&1&-2\\-\frac{1}{2}&-\frac{1}{2}&\frac{5}{2}\end{bmatrix}$$

tražena inverzna matrica

③ Naći inverznu matricu matrice $C=\begin{bmatrix}2&1\\5&4\end{bmatrix}$.

Rj. $C^{-1}=\dfrac{1}{\det C}C_{kof}^{T}$, $\det C=\begin{vmatrix}2&1\\5&4\end{vmatrix}=3$

$C_{11}=(-1)^2\cdot 4=4$ $\quad C_{21}=(-1)^3\cdot 1=-1$

$C_{12}=(-1)^3\cdot 5=-5$ $\quad C_{22}=(-1)^4\cdot 2=2$

$$C_{kof}^{T}=\begin{bmatrix}4&-1\\-5&2\end{bmatrix}\quad C^{-1}=\begin{bmatrix}\frac{4}{3}&-\frac{1}{3}\\-\frac{5}{3}&\frac{2}{3}\end{bmatrix}$$

④ Naći inverznu matricu sljedećih matrica:

a) $A=\begin{bmatrix}3&4&4\\1&6&1\\2&3&3\end{bmatrix}$

b) $B=\begin{bmatrix}-3&-1&-1\\1&3&2\\-2&-1&-2\end{bmatrix}$

c) $C=\begin{bmatrix}7&3&3\\6&3&4\\-1&-2&-3\end{bmatrix}$

Rješenja:

a) $A^{-1}=\begin{bmatrix}3&0&-4\\-\frac{1}{5}&\frac{1}{5}&\frac{1}{5}\\-\frac{8}{5}&-\frac{1}{5}&\frac{14}{5}\end{bmatrix}$

b) $B^{-1}=\begin{bmatrix}-\frac{4}{9}&-\frac{1}{9}&\frac{1}{9}\\-\frac{2}{9}&\frac{4}{9}&\frac{5}{9}\\\frac{5}{9}&-\frac{1}{9}&-\frac{8}{9}\end{bmatrix}$

c) $\det C=8$

Matrične jednačine

U sljedecim primjerima neka su A, B, C, X neke date kvadratne matrice.

Da bismo odredili nepoznatu X u matričnoj jednačini prvo trebamo izvesti formulu za nepoznatu X.

$$A^{-1}\cdot B \neq B\cdot A^{-1}$$

$$A\cdot B \neq B\cdot A$$

Matrice se ne mogu dijeliti.

$\#$ $A\cdot X = B$ $\quad /\cdot A^{-1}$ sa lijeve strane

$$A^{-1}\cdot A\cdot X = A^{-1}\cdot B$$

$$1\cdot X = A^{-1}\cdot B$$

$$X = A^{-1}\cdot B$$

$\#$ $A\cdot X\cdot B = C$ $\quad /\cdot A^{-1}$ sa lijeve strane

$$A^{-1}\cdot A\cdot X\cdot B = A^{-1}\cdot C$$

$1\cdot X\cdot B = A^{-1}\cdot C$ $\quad /B^{-1}$ sa desne strane

$$X\cdot B\cdot B^{-1} = A^{-1}\cdot C\cdot B^{-1}$$

$$X\cdot 1 = A^{-1}\cdot C\cdot B^{-1}$$

$$X = A^{-1}\cdot C\cdot B^{-1}$$

$\#$ $A\cdot X + 1 = X - 21$

$$AX - X = -1 - 21$$

$$\underbrace{(A-1)}_{B} X = -31$$

$BX = -31$ $\quad /\cdot B^{-1}$ sa desne strane

$$B^{-1}\cdot B\cdot X = B^{-1}\cdot(-31)$$

$$1\cdot X = -3B^{-1}$$

$$X = -3(A-1)^{-1}$$

$\#$ $X^{-1}\cdot A = B^{-1}$ $\quad /\cdot A^{-1}$ sa desne strane

$$X^{-1}\cdot A\cdot A^{-1} = B^{-1}\cdot A^{-1}$$

$$X^{-1}\cdot 1 = B^{-1}\cdot A^{-1}$$

$X^{-1} = B^{-1}\cdot A^{-1}$ $\quad /^{(-1)}$

$$X = A\cdot B$$

$\#$ $A^{-1}\cdot X = X - 1$

$$A^{-1}\cdot X - X = -1$$

$$\underbrace{(A^{-1} - 1)}_{B}\cdot X = -1$$

$BX = -1$ $\quad /\cdot B^{-1}$ sa lijeve strane

$$B^{-1}\cdot B\cdot X = -B^{-1}\cdot 1$$

$$X = -B^{-1}$$

$$X = -(A^{-1}-1)^{-1}$$

\#

$$\underbrace{(A+3I)}_{C}(X-I)=B$$

$C(X-I)=B \quad /\cdot C^{-1}$ sa lijeve strane

$$C^{-1}C(X-I)=C^{-1}\cdot B$$

$$X-I=C^{-1}\cdot B$$

$$X=C^{-1}\cdot B+I$$

$$X=(A+3I)^{-1}\cdot B+I$$

\#

$B^{-1}\cdot X\cdot A=(3B+2I)^{-1}$

$/\cdot B$ sa lijeve strane

$$B\cdot B^{-1}\cdot X\cdot A=B(3B+2\cdot I)^{-1}$$

$X\cdot A=B(3B+2I)^{-1} \quad /\cdot A^{-1}$ sa desne strane

$$\underline{X=B(3B+2I)^{-1}\cdot A^{-1}}$$

\#

$(AXB)^{-1}=B^{-1}(X^{-1}+B) \quad /\cdot(AXB)$ sa lijeve strane

$$(AXB)(AXB)^{-1}=AX\underbrace{BB^{-1}}_{I}(X^{-1}+B)$$

$$I=AX(X^{-1}+B)$$

$$I=AXX^{-1}+AXB$$

$$I=A+AXB$$

$AXB=I-A \quad /\cdot A^{-1}$ sa lijeve str. $/\cdot B^{-1}$ sa desne str.

$$A^{-1}AXB\cdot B^{-1}=A^{-1}(I-A)\cdot B^{-1}$$

$$X=A^{-1}(I-A)\cdot B^{-1}$$

① Riješiti matričnu jednačinu $X\cdot\begin{bmatrix}1&1&1\\-1&-2&-3\\2&3&5\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix}$.

Rj. $X\cdot A=B \quad /\cdot A^{-1}$ sa desne str.

$X=B\cdot A^{-1}$, $\quad A=\begin{bmatrix}1&1&1\\-1&-2&-3\\2&3&5\end{bmatrix}$

$$A^{-1}=\frac{1}{\det A}\cdot A_{kof}^{T}$$

$A_{11}=(-1)^2\begin{vmatrix}-2&-3\\3&5\end{vmatrix}=-1$

$A_{12}=(-1)^3\begin{vmatrix}-1&-3\\2&5\end{vmatrix}=-1$

$A_{13}=(-1)^4\begin{vmatrix}-1&-2\\2&3\end{vmatrix}=1$

$$\det A=\begin{vmatrix}1&1&1\\-1&-2&-3\\2&3&5\end{vmatrix}\underset{II_k-III_k}{\overset{I_k-III_k}{=\!=}}\begin{vmatrix}0&0&1\\2&1&-3\\-3&-2&5\end{vmatrix}=\begin{vmatrix}2&1\\-3&-2\end{vmatrix}=-1.$$

$A_{21}=(-1)^3\begin{vmatrix}1&1\\3&5\end{vmatrix}=-2$

$A_{22}=(-1)^4\begin{vmatrix}1&1\\2&5\end{vmatrix}=3$

$A_{23}=(-1)^5\begin{vmatrix}1&1\\2&3\end{vmatrix}=-1$

$A_{31}=(-1)^4\begin{vmatrix}1&1\\-2&-3\end{vmatrix}=-1$

$A_{32}=(-1)^5\begin{vmatrix}1&1\\-1&-3\end{vmatrix}=2$

$A_{33}=(-1)^6\begin{vmatrix}1&1\\-1&-2\end{vmatrix}=-1$

$$A_{kof}^{T}=\begin{bmatrix}-1&-2&-1\\-1&3&2\\1&-1&-1\end{bmatrix}\qquad A^{-1}=\begin{bmatrix}1&2&1\\1&-3&-2\\-1&1&1\end{bmatrix}$$

$$X = B\cdot A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & -3 & -2 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -2 \\ 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$ rješenje matrične jednačine

②. Riješiti matričnu jednačinu $A\cdot X = X + I$ ako je $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 3 & -2 \\ 3 & 1 & -1 \end{bmatrix}$.

Rj.

$$AX = X + I$$

$$C = A - I = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 3 & -2 \\ 3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -2 \\ 3 & 1 & -2 \end{bmatrix}$$

$$AX - X = I$$

$$(A-I)X = I \quad /\cdot(A-I)^{-1}$$ sa lijeve strane

$$C^{-1} = \frac{1}{\det C}\, C_{kof}^{T}$$

$$(A-I)(A-I)^{-1}X = (A-I)^{-1}\cdot I$$

$$X = (A-I)^{-1}$$

$$\det C = \begin{vmatrix} 0 & -1 & 1 \\ 1 & 2 & -2 \\ 3 & 1 & -2 \end{vmatrix} \xlongequal{II_k + III_k} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 3 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -1$$

$$C_{11} = (-1)^2 \begin{vmatrix} 2 & -2 \\ 1 & -2 \end{vmatrix} = -2 \qquad C_{21} = (-1)^3 \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} = -1 \qquad C_{31} = (-1)^4 \begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix} = 0$$

$$C_{12} = (-1)^3 \begin{vmatrix} 1 & -2 \\ 3 & -2 \end{vmatrix} = -4 \qquad C_{22} = (-1)^4 \begin{vmatrix} 0 & 1 \\ 3 & -2 \end{vmatrix} = -3 \qquad C_{32} = (-1)^5 \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = 1$$

$$C_{13} = (-1)^4 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5 \qquad C_{23} = (-1)^5 \begin{vmatrix} 0 & -1 \\ 3 & 1 \end{vmatrix} = -3 \qquad C_{33} = (-1)^6 \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = 1$$

$$C_{kof}^{T} = \begin{bmatrix} -2 & -1 & 0 \\ -4 & -3 & 1 \\ -5 & -3 & 1 \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & -1 \\ 5 & 3 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & -1 \\ 5 & 3 & -1 \end{bmatrix}$$ rješenje

③. Riješiti matričnu jednačinu $(A+B)^{-1}AX^{-1} = B^{-1}$ gdje su matrice $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ i $B = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$.

Rj.

$$(A+B)^{-1}A\cdot X^{-1} = B^{-1} \quad /\cdot(A+B)$$ sa lijeve strane

$$(A+B)(A+B)^{-1}\cdot A\cdot X^{-1} = (A+B)\cdot B^{-1}$$

$$A\cdot X^{-1} = (A+B)\cdot B^{-1} \quad /\cdot A^{-1}$$ sa lijeve strane

$$A^{-1}\cdot A\cdot X^{-1} = A^{-1}(A+B)\cdot B^{-1}$$

$$X^{-1} = A^{-1}(A+B)\cdot B^{-1} \quad /^{-1}$$

$$X = B(A+B)^{-1}A$$

$$C = A+B = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}$$

$C=\begin{bmatrix}4&-1\\1&3\end{bmatrix}$, $C^{-1}=\frac{1}{\det C}C_{kof}^{T}$, $\det C=\begin{vmatrix}4&-1\\1&3\end{vmatrix}=13$,

$C_{11}=(-1)^2\cdot 3=3$
$C_{12}=(-1)^3\cdot 1=-1$
$C_{21}=(-1)^3\cdot(-1)=1$
$C_{22}=(-1)^4\cdot 4=4$

$C_{kof}^{T}=\begin{bmatrix}3&1\\-1&4\end{bmatrix}$, $C^{-1}=\begin{bmatrix}\frac{3}{13}&\frac{1}{13}\\-\frac{1}{13}&\frac{4}{13}\end{bmatrix}$

$$X=B\cdot C^{-1}\cdot A=\begin{bmatrix}3&0\\-1&2\end{bmatrix}\cdot\frac{1}{13}\cdot\begin{bmatrix}3&1\\-1&4\end{bmatrix}\cdot\begin{bmatrix}1&-1\\2&1\end{bmatrix}=\frac{1}{13}\begin{bmatrix}3&0\\-1&2\end{bmatrix}\begin{bmatrix}5&-2\\7&5\end{bmatrix}=\frac{1}{13}\begin{bmatrix}15&-6\\9&12\end{bmatrix}$$

$X=\begin{bmatrix}\frac{15}{13}&-\frac{8}{13}\\\frac{9}{13}&\frac{12}{13}\end{bmatrix}$ rješenje matrične jednačine

④ Riješiti matričnu jednačinu $(A+3I)(X-I)=B$, ako je $A=\begin{bmatrix}-2&5&-2\\2&8&0\\-1&-5&-2\end{bmatrix}$, $B=\begin{bmatrix}-3&21&1\\2&50&-2\\1&-22&0\end{bmatrix}$ i I jedinična matrica.

Rj. $(A+3I)(X-I)=B$ $\quad/\cdot(A+3I)^{-1}$ sa lijeve strane

$(A+3I)^{-1}(A+3I)(X-I)=(A+3I)^{-1}\cdot B$

$X-I=(A+3I)^{-1}\cdot B$

$X=(A+3I)^{-1}\cdot B+I$

$C^{-1}=\frac{1}{\det C}C_{kof}^{T}$

$\det C=\begin{vmatrix}1&5&-2\\2&11&0\\-1&-5&1\end{vmatrix}\ \underline{\underline{I_R+III_R}}$

$=\begin{vmatrix}0&0&-1\\2&11&0\\-1&-5&1\end{vmatrix}=(-1)\begin{vmatrix}2&11\\-1&-5\end{vmatrix}=-1$

$$C=A+3I=\begin{bmatrix}-2&5&-2\\2&8&0\\-1&-5&-2\end{bmatrix}+\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}=\begin{bmatrix}1&5&-2\\2&11&0\\-1&-5&1\end{bmatrix}$$

$C_{11}=(-1)^2\begin{vmatrix}11&0\\-5&1\end{vmatrix}=11$

$C_{12}=(-1)^3\begin{vmatrix}2&0\\-1&1\end{vmatrix}=-2$

$C_{13}=(-1)^4\begin{vmatrix}2&11\\-1&-5\end{vmatrix}=1$

$C_{21}=(-1)^3\begin{vmatrix}5&-2\\-5&1\end{vmatrix}=5$ $\qquad C_{31}=(-1)^4\begin{vmatrix}5&-2\\11&0\end{vmatrix}=22$

$C_{22}=(-1)^4\begin{vmatrix}1&-2\\-1&1\end{vmatrix}=-1$ $\qquad C_{32}=(-1)^5\begin{vmatrix}1&-2\\2&0\end{vmatrix}=-4$

$C_{23}=(-1)^5\begin{vmatrix}1&5\\-1&-5\end{vmatrix}=0$ $\qquad C_{33}=(-1)^6\begin{vmatrix}1&5\\2&11\end{vmatrix}=1$ $\qquad C_{kof}^{T}=\begin{bmatrix}11&5&22\\-2&-1&-4\\1&0&1\end{bmatrix}$

$C^{-1}=\begin{bmatrix}-11&-5&-22\\2&1&4\\-1&0&-1\end{bmatrix}$ $\qquad C^{-1}\cdot B=\begin{bmatrix}-11&-5&-22\\2&1&4\\-1&0&-1\end{bmatrix}\cdot\begin{bmatrix}-3&21&1\\2&50&-2\\1&-22&0\end{bmatrix}=\begin{bmatrix}1&3&-1\\0&4&0\\2&1&-1\end{bmatrix}$

$$X=(A+3I)^{-1}\cdot B+I=\begin{bmatrix}1&3&-1\\0&4&0\\2&1&-1\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}2&3&-1\\0&5&0\\2&1&0\end{bmatrix}$$ rješenje matrične jednačine

5) Riješiti matričnu jednačinu $(X^{-1}+B^{-1})^{-1}=AX$ ako su

$$A=\begin{bmatrix}3&3&2\\-4&1&-4\\-3&1&-3\end{bmatrix} \quad i \quad B=\begin{bmatrix}1&0&2\\1&-2&0\\0&-1&1\end{bmatrix}.$$

Rj. $(X^{-1}+B^{-1})^{-1}=AX \quad /\cdot(X^{-1}+B^{-1})$ sa desne strane

$$(X^{-1}+B^{-1})^{-1}\cdot(X^{-1}+B^{-1})=AX(X^{-1}+B^{-1})$$

$$I=A+AXB^{-1}$$

$AXB^{-1}=I-A \quad /\cdot A^{-1}$ sa lijeve str. $\cdot B$ sa desne str.

$$A^{-1}\cdot A\cdot X\cdot B^{-1}\cdot B=A^{-1}(I-A)\cdot B$$

$$X=A^{-1}(I-A)\cdot B$$

$$A^{-1}=\frac{1}{\det A}\cdot A_{kof}^{-1}$$

$$\det A=\begin{vmatrix}3&3&2\\-4&1&-4\\-3&1&-3\end{vmatrix}\underline{\underline{I_k-III_k}}$$

$$=\begin{vmatrix}1&3&2\\0&1&-4\\0&1&-3\end{vmatrix}=\begin{vmatrix}1&-4\\1&-3\end{vmatrix}$$

$$=-3+4=1$$

$$A_{11}=(-1)^2\begin{vmatrix}1&-4\\1&-3\end{vmatrix}=1$$

$$A_{12}=(-1)^3\begin{vmatrix}-4&-4\\-3&-3\end{vmatrix}=0$$

$$A_{13}=(-1)^4\begin{vmatrix}-4&1\\-3&1\end{vmatrix}=-1$$

$$A_{21}=(-1)^3\begin{vmatrix}3&2\\1&-3\end{vmatrix}=11 \qquad A_{31}=(-1)^4\begin{vmatrix}3&2\\1&-4\end{vmatrix}=-14$$

$$A_{22}=(-1)^4\begin{vmatrix}3&2\\-3&-3\end{vmatrix}=-3 \qquad A_{32}=(-1)^5\begin{vmatrix}3&2\\-4&-4\end{vmatrix}=4$$

$$A_{23}=(-1)^5\begin{vmatrix}3&3\\-3&1\end{vmatrix}=-12 \qquad A_{33}=(-1)^6\begin{vmatrix}3&3\\-4&1\end{vmatrix}=15$$

$$A_{kof}=\begin{bmatrix}1&0&-1\\11&-3&-12\\-14&4&15\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}1&11&-14\\0&-3&4\\-1&-12&15\end{bmatrix}, \quad X=A^{-1}(I-A)\cdot B=\begin{bmatrix}1&11&-14\\0&-3&4\\-1&-12&15\end{bmatrix}\begin{bmatrix}-2&-3&-2\\4&0&4\\3&-1&4\end{bmatrix}\begin{bmatrix}1&0&2\\1&-2&0\\0&-1&1\end{bmatrix}$$

$$=\begin{bmatrix}1&11&-14\\0&-3&4\\-1&-12&15\end{bmatrix}\begin{bmatrix}-5&8&-6\\4&-4&12\\2&-2&10\end{bmatrix}=\begin{bmatrix}11&-8&-14\\-4&4&4\\-13&10&12\end{bmatrix}$$ rješenje matrične jednačine

6) Riješiti matričnu jednačinu:

$$X\cdot\begin{bmatrix}2&3&4\\1&2&3\\4&3&2\end{bmatrix}+\begin{bmatrix}1&4&5\\0&1&4\\3&4&2\end{bmatrix}=X\cdot\begin{bmatrix}1&4&5\\0&1&4\\3&4&2\end{bmatrix}.$$

Ako označimo $A=\begin{bmatrix}2&3&4\\1&2&3\\4&3&2\end{bmatrix}$ i $B=\begin{bmatrix}1&4&5\\0&1&4\\3&4&2\end{bmatrix}$ imamo

$$XA+B=XB$$

$$XA-XB=-B$$

$$X(A-B)=-B \quad /\cdot(A-B)^{-1}$$ sa desne strane

$$X=-B(A-B)^{-1}$$

$$\det C=\begin{vmatrix}1&-1&-1\\1&1&-1\\1&-1&0\end{vmatrix}\xlongequal{I_k+II_k}\begin{vmatrix}0&-1&-1\\2&1&-1\\0&-1&0\end{vmatrix}$$

$$=(-2)\begin{vmatrix}-1&-1\\-1&0\end{vmatrix}=(-2)\cdot(-1)=2$$

$$C=A-B=\begin{bmatrix}1&-1&-1\\1&1&-1\\1&-1&0\end{bmatrix}$$

$$C_{11}=(-1)^2\begin{vmatrix}1&-1\\-1&0\end{vmatrix}=-1 \qquad C_{21}=1 \qquad C_{31}=2$$

$$C_{12}=(-1)^3\cdot 1=-1 \qquad C_{22}=1 \qquad C_{32}=0$$

$$C_{13}=-2 \qquad C_{23}=0 \qquad C_{33}=2$$

$$C^{-1}=\frac{1}{\det C}\cdot C_{kof}^{T}$$

$$C^{-1}=\frac{1}{2}\begin{bmatrix}-1&1&2\\-1&1&0\\-2&0&2\end{bmatrix}, \qquad X=-B\cdot C^{-1}=-\begin{bmatrix}1&4&5\\0&1&4\\3&4&2\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}-1&1&2\\-1&1&0\\-2&0&2\end{bmatrix}$$

$$X=-\frac{1}{2}\begin{bmatrix}-15&5&12\\-9&1&8\\-11&7&10\end{bmatrix}=\begin{bmatrix}\frac{15}{2}&-\frac{5}{2}&-6\\\frac{9}{2}&-\frac{1}{2}&-4\\\frac{11}{2}&-\frac{7}{2}&-5\end{bmatrix}$$ rješenje matrične jednačine

7. Riješiti matričnu jednačinu $(A+X)(B-2I)=A$, ako su $A=\begin{bmatrix}-2&-3&-4\\1&-2&3\\4&3&-2\end{bmatrix}$, $B=\begin{bmatrix}3&-2&1\\1&-1&0\\1&-2&2\end{bmatrix}$, I jedinična matrica.

8. Riješiti matričnu jednačinu $A^{-1}X+B=AX$, ako su $A=\begin{bmatrix}-2&5\\-1&3\end{bmatrix}$ i $B=\begin{bmatrix}4&5\\6&7\end{bmatrix}$.

9. Riješiti matričnu jednačinu $(XB^{-1})^{-1}=X^{-1}+A$, ako su $A=\begin{bmatrix}-1&3&1\\1&2&3\\0&3&2\end{bmatrix}$; $B=\begin{bmatrix}1&1&1\\5&5&4\\3&2&1\end{bmatrix}$.

Rješenja:

7. $X=\begin{bmatrix}-2&10&-1\\2&2&-5\\-6&-14&19\end{bmatrix}$

8. $X=\begin{bmatrix}4&5\\6&7\end{bmatrix}$

9. $X=\begin{bmatrix}3&-\frac{3}{2}&-\frac{17}{2}\\1&-1&-5\\0&\frac{5}{2}&\frac{15}{2}\end{bmatrix}$

Ⓣ Data je matrična jednačina $A(X-B)^{-1}=B^{-1}A$ i matrice

$$A=\begin{bmatrix}1&1&1\\0&1&1\\0&-1&1\end{bmatrix} \quad i \quad B=\begin{bmatrix}1&1&0\\2&1&0\\3&-1&2\end{bmatrix}.$$

a) Koji uslov moraju zadovoljavat matrice A i B da bi data jednačina imala rješenje $X=2B$?

b) Riješiti datu jednačinu ako matrice A i B ne zadovoljavaju uslov dobijen pod a)

Rj. a) $A(X-B)^{-1}=B^{-1}A$

$X=2B$

$A\cdot B^{-1}=B^{-1}A$ uslov koji moraju zadovoljavati matrice A i B da bi data matrica imala rješenje $X=2B$.

Uslov možemo pisati i na drugi način:

$A=B^{-1}AB$

ili

$B=A^{-1}\cdot B\cdot A$

b)

$A(X-B)^{-1}=B^{-1}\cdot A \quad /\cdot(X-B)$ sa desne str

$B^{-1}A(X-B)=A \quad /\cdot B$ sa lijeve str.

$A(X-B)=BA \quad /\cdot A^{-1}$ sa lijeve str.

$X-B=A^{-1}BA$

$X=A^{-1}BA+B$

i odavde možemo pročitati uslov koji smo dobili pod a) (ako je $B=A^{-1}BA$ tada jednačina ima rješenje $X=2B$)

Provjerimo da li je $B=A^{-1}BA$.

Nađimo prvo A^{-1}

$$A^{-1}=\frac{1}{\det A}\cdot A_{kof}^{T}$$

$$\det A=\begin{vmatrix}1&1&1\\0&1&1\\0&-1&1\end{vmatrix}=\begin{vmatrix}1&1\\-1&1\end{vmatrix}=2$$

$A_{11}=(-1)^2\begin{vmatrix}1&1\\-1&1\end{vmatrix}=2 \qquad A_{21}=(-1)^2\begin{vmatrix}1&1\\-1&1\end{vmatrix}=-2 \qquad A_{31}=(-1)^4\begin{vmatrix}1&1\\1&1\end{vmatrix}=0$

$A_{12}=(-1)^3\begin{vmatrix}0&1\\0&1\end{vmatrix}=0 \qquad A_{22}=(-1)^4\begin{vmatrix}1&1\\0&1\end{vmatrix}=1 \qquad A_{32}=(-1)^5\begin{vmatrix}1&1\\0&1\end{vmatrix}=-1$

$A_{13}=(-1)^4\begin{vmatrix}0&1\\0&-1\end{vmatrix}=0 \qquad A_{23}=(-1)^5\begin{vmatrix}1&1\\0&-1\end{vmatrix}=1 \qquad A_{33}=(-1)^6\begin{vmatrix}1&1\\0&1\end{vmatrix}=1$

$$A_{kof}=\begin{bmatrix}2&0&0\\-2&1&1\\0&-1&1\end{bmatrix} \qquad A_{kof}^{T}=\begin{bmatrix}2&-2&0\\0&1&-1\\0&1&1\end{bmatrix}$$

$$A^{-1}=2\begin{bmatrix}2&-2&0\\0&1&-1\\0&1&1\end{bmatrix}, \quad B\cdot A=\begin{bmatrix}1&1&0\\2&1&0\\3&-1&2\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&1\\0&-1&1\end{bmatrix}=\begin{bmatrix}1&2&2\\2&3&3\\3&0&4\end{bmatrix}$$

$$A^{-1}\cdot B\cdot A=2\begin{bmatrix}2&-2&0\\0&1&-1\\0&1&1\end{bmatrix}\begin{bmatrix}1&2&2\\2&3&3\\3&0&4\end{bmatrix}=2\begin{bmatrix}-2&-2&-2\\-1&3&-1\\5&3&7\end{bmatrix}=\begin{bmatrix}-1&-1&-1\\-\frac{1}{2}&\frac{3}{2}&-\frac{1}{2}\\\frac{5}{2}&\frac{3}{2}&\frac{7}{2}\end{bmatrix}$$

ovdje vidimo da matrice A i B ne zadovoljavaju uslov dobijen pod a)

$$X=A^{-1}BA+B=\begin{bmatrix}-1&-1&-1\\-\frac{1}{2}&3/2&-1/2\\5/2&3/2&7/2\end{bmatrix}+\begin{bmatrix}1&1&0\\2&1&0\\3&-1&2\end{bmatrix}=\begin{bmatrix}0&0&-1\\3/2&5/2&-1/2\\11/2&1/2&11/2\end{bmatrix}$$ rješenje matrične jednačine

(#) Riješiti matričnu jednačinu $X \cdot A^{-1} = B^{-1}$ ako su

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & -4 \end{bmatrix} \quad i \quad B = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}.$$

Rj.

$X \cdot A^{-1} = B^{-1}$ $/ \cdot A$ sa desne strane

$$X \underbrace{A^{-1} \cdot A}_{I} = B^{-1} \cdot A$$

$$X = B^{-1} \cdot A$$

$$\det B = \begin{vmatrix} 2 & 1 & -1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{vmatrix} \xlongequal{I_R - II_R}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = 0 + 1$$

$$\det B = 1$$

$$B^{-1} = \frac{1}{\det B} B_{kof}^{T}$$

$$B_{11} = (-1)^2 \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1 \qquad B_{21} = (-1)^3 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = -1 \qquad B_{31} = (-1)^4 \begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} = -1$$

$$B_{12} = (-1)^3 \begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} = 0 \qquad B_{22} = (-1)^4 \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = 1 \qquad B_{32} = (-1)^5 \begin{vmatrix} 2 & -1 \\ 2 & -2 \end{vmatrix} = 2$$

$$B_{13} = (-1)^4 \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = 1 \qquad B_{23} = (-1)^5 \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = -1 \qquad B_{33} = (-1)^6 \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0$$

$$B_{kof} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ -1 & 2 & 0 \end{bmatrix}, \quad B_{kof}^{T} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$X = B^{-1} \cdot A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & -4 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 7 \\ 5 & 4 & -7 \\ -1 & 1 & 3 \end{bmatrix}$$

$2-3-1 \qquad 0+3+2 \qquad 2-3+0$

$3-2-1 \qquad 0+2+2 \qquad 3-2+0$

$4-1+4 \qquad 0+1-8 \qquad 4-1+0$

$$X = \begin{bmatrix} -2 & 0 & 7 \\ 5 & 4 & -7 \\ -1 & 1 & 3 \end{bmatrix}$$

traženo rješenje

(#) Riješiti matričnu jednačinu $X^{-1}AB = B^{-1}A^{-1}$,

$$A = \begin{bmatrix} 1 & 1 & 6 \\ 2 & -3 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -4 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

Rj.

$$X^{-1}AB = B^{-1}A^{-1}$$

$$X^{-1}AB = (AB)^{-1} \quad /(AB)^{-1} \text{ sa desne strane}$$

$$X^{-1} = (AB)^{-1}(AB)^{-1}$$

$$X = (AB)\cdot(AB)$$

$$X = (AB)^2$$

$$A\cdot B = \begin{bmatrix} 1 & 1 & 6 \\ 2 & -3 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ 1 & -4 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & -5 & 13 \\ 1 & 10 & -3 \\ 2 & -4 & 3 \end{bmatrix}$$

$2+1+6$ $\quad$ $4-3+0$ $\quad$ $0+1+1$

$-1-4+0$ $\quad$ $-2+12+0$ $\quad$ $0-4+0$

$0+1+12$ $\quad$ $0-3+0$ $\quad$ $0+1+2$

$$(AB)^2 = \begin{bmatrix} 9 & -5 & 13 \\ 1 & 10 & -3 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 9 & -5 & 13 \\ 1 & 10 & -3 \\ 2 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 102 & -147 & 171 \\ 13 & 107 & -26 \\ 20 & -62 & 47 \end{bmatrix}$$

$81-5+26$ $\quad$ $-45-50-52$ $\quad$ $117+15+39$

$9+10-6$ $\quad$ $-5+100+12$ $\quad$ $13-30-9$

$18-4+6$ $\quad$ $-10-40-12$ $\quad$ $26+12+9$

$$X = \begin{bmatrix} 102 & -147 & 171 \\ 12 & 107 & -26 \\ 20 & -62 & 47 \end{bmatrix}$$

(#) Riješiti matričnu jednačinu $(A+I)^{-1}\cdot X\cdot(3A+I)=2A$ gdje je I jedinična matrica drugog reda a

$$A=\begin{bmatrix}7 & 8\\ -6 & -7\end{bmatrix}.$$

Rj. $(A+I)^{-1}\cdot X\cdot(3A+I)=2A \quad /(A+I)$ sa lijeve strane

$X\cdot(3A+I)=(A+I)\cdot 2A \quad /\cdot(3A+I)^{-1}$ sa desne strane

$X=(A+I)\cdot 2A\cdot(3A+I)^{-1}$

$$A=\begin{bmatrix}7 & 8\\ -6 & -7\end{bmatrix} \qquad I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad A+I=\begin{bmatrix}8 & 8\\ -6 & -6\end{bmatrix}$$

20·22
40
40
440

$$3A+I=\begin{bmatrix}22 & 24\\ -18 & -20\end{bmatrix} \qquad 3A=\begin{bmatrix}21 & 24\\ -18 & -21\end{bmatrix}$$

18·24
72
36
432

Označimo sa $B=3A-1$ pa pronađimo B^{-1}

$$B^{-1}=\frac{1}{\det B}\cdot B_{kof}^{T} \qquad \det B=\begin{vmatrix}22 & 24\\ -18 & -20\end{vmatrix}=-440+432=-8$$

$B_{11}=(-1)^2\cdot(-20)=-20$ $\quad B_{21}=(-1)^3\cdot 24=-24$

$B_{12}=(-1)^3\cdot(-18)=18$ $\quad B_{22}=(-1)^4\cdot 22=22$

$$B_{kof}=\begin{bmatrix}-20 & 18\\ -24 & 22\end{bmatrix}$$

$$B^{-1}=\frac{-1}{8}\begin{bmatrix}-20 & -24\\ 18 & 22\end{bmatrix}=(3A+I)^{-1}$$

$$X=(A+I)\cdot 2A\cdot(3A+I)^{-1}=\begin{bmatrix}8 & 8\\ -6 & -6\end{bmatrix}\cdot 2\cdot\begin{bmatrix}7 & 8\\ -6 & -7\end{bmatrix}\cdot\frac{-1}{8}\begin{bmatrix}-20 & -24\\ 18 & 22\end{bmatrix}$$

$$=2\begin{bmatrix}4 & 4\\ -3 & -3\end{bmatrix}\cdot 2\begin{bmatrix}7 & 8\\ -6 & -7\end{bmatrix}\cdot\frac{-1}{8}\cdot 2\begin{bmatrix}-10 & -12\\ 9 & 11\end{bmatrix}=8\cdot\frac{-1}{8}\begin{bmatrix}4 & 4\\ -3 & -3\end{bmatrix}\begin{bmatrix}7 & 8\\ -6 & -7\end{bmatrix}\begin{bmatrix}-10 & -12\\ 9 & 11\end{bmatrix}$$

$$=(-1)\begin{bmatrix}4 & 4\\ -3 & -3\end{bmatrix}\begin{bmatrix}-10 & -12\\ 9 & 11\end{bmatrix}=(-1)\begin{bmatrix}-4 & -4\\ 3 & 3\end{bmatrix}=\begin{bmatrix}4 & 4\\ -3 & -3\end{bmatrix}$$ rješenje matrične jednačine

(#) Riješiti matričnu jednačinu $(AXB)^{-1} = B^{-1}(X^{-1}+B)$

ako je $A = \begin{bmatrix}3 & -4 & 5\\ 2 & -3 & 1\\ 3 & -5 & -1\end{bmatrix}$, $B = \begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$.

Rj.

$$(AXB)^{-1} = B^{-1}(X^{-1}+B)$$

$$B^{-1}X^{-1}A^{-1} = B^{-1}X^{-1} + B^{-1}\cdot B \quad /\cdot B \text{ sa lijeve strane}$$

$$X^{-1}A^{-1} = X^{-1} + B$$

$$X^{-1}A^{-1} - X^{-1} = B$$

$$X^{-1}(A^{-1}-I) = B \quad /\cdot(A^{-1}-I)^{-1} \text{ sa desne strane}$$

$$X^{-1} = B(A^{-1}-I)^{-1} \quad /^{-1}$$

$$X = (A^{-1}-I)\cdot B^{-1}$$

$$\det A = \begin{vmatrix}3 & -4 & 5\\ 2 & -3 & 1\\ 3 & -5 & -1\end{vmatrix} \overset{I_k+II_k}{=\!=} \begin{vmatrix}-1 & -4 & 5\\ -1 & -3 & 1\\ -2 & -5 & -1\end{vmatrix} \underset{III_v - I_v\cdot 2}{\overset{II_v - I_v}{=\!=}}$$

$$= \begin{vmatrix}-1 & -4 & 5\\ 0 & 1 & -4\\ 0 & 3 & -11\end{vmatrix} = (-1)\begin{vmatrix}1 & -4\\ 3 & -11\end{vmatrix} = (-1)(-11+12) = -1$$

$$A^{-1} = \frac{1}{\det A}\cdot A_{kof}^{T}$$

$A_{11} = (-1)^2\begin{vmatrix}-3 & 1\\ -5 & -1\end{vmatrix} = 3+5 = 8$

$A_{12} = (-1)^3\begin{vmatrix}2 & 1\\ 3 & -1\end{vmatrix} = -(-2-3) = 5$

$A_{13} = (-1)^4\begin{vmatrix}2 & -3\\ 3 & -5\end{vmatrix} = -10+9 = -1$

$A_{21} = (-1)^3\begin{vmatrix}-4 & 5\\ -5 & -1\end{vmatrix} = -(4+25) = -29$

$A_{22} = (-1)^4\begin{vmatrix}3 & 5\\ 3 & -1\end{vmatrix} = -3-15 = -18$

$A_{23} = (-1)^5\begin{vmatrix}3 & -4\\ 3 & -5\end{vmatrix} = -(-15+12) = 3$

$A_{31} = 11$

$A_{32} = 7$

$A_{33} = -1$

$$A_{kof} = \begin{bmatrix}8 & 5 & -1\\ -29 & -18 & 3\\ 11 & 7 & -1\end{bmatrix}. \quad A^{-1} = (-1)\begin{bmatrix}8 & -29 & 11\\ 5 & -18 & 7\\ -1 & 3 & -1\end{bmatrix} = \begin{bmatrix}-8 & 29 & -11\\ -5 & 18 & -7\\ 1 & -3 & 1\end{bmatrix}.$$

$$\det B = \begin{vmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{vmatrix} \underset{III_v - I_v\cdot 2}{\overset{II_v - I_v\cdot 2}{=\!=}} \begin{vmatrix}1 & 2 & 2\\ 0 & -3 & -6\\ 0 & -6 & -3\end{vmatrix} = \begin{vmatrix}-3 & -6\\ -6 & -3\end{vmatrix} = 9-36 = -27$$

$$B^{-1} = \frac{1}{\det B}\cdot B_{kof}^{T} =\!= \frac{(-1)}{-27}\begin{bmatrix}3 & 6 & 6\\ 6 & 3 & -6\\ 6 & -6 & 3\end{bmatrix} = \frac{1}{27}\cdot 3\cdot\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$$

ZA VJEŽBU (slično)

$$B^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$A^{-1} - I = \begin{bmatrix} -8 & 29 & -11 \\ -5 & 18 & -7 \\ 1 & -3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 29 & -11 \\ -5 & 17 & -7 \\ 1 & -3 & 0 \end{bmatrix}$$

$$X = (A^{-1} - I) \cdot B^{-1} = \begin{bmatrix} -9 & 29 & -11 \\ -5 & 17 & -7 \\ 1 & -3 & 0 \end{bmatrix} \cdot \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} =$$

$$= \frac{1}{9}\begin{bmatrix} 27 & 33 & -87 \\ 15 & 21 & -51 \\ -5 & -1 & 8 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 & 11/3 & -\frac{29}{3} \\ 5/3 & 7/3 & -\frac{17}{3} \\ -\frac{5}{9} & -\frac{1}{9} & \frac{8}{9} \end{bmatrix}$$ rješenje matrične jednačine

(#) Riješiti matričnu jednačinu $A \cdot X^{-1} \cdot B = B \cdot A$, ako je $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ i $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Rj.

$A X^{-1} B = B \cdot A$ $\quad / \cdot A^{-1}$ sa lijeve strane

$X^{-1} B = A^{-1} B \cdot A$ $\quad / \cdot B^{-1}$ sa desne strane

$X^{-1} = A^{-1} B \cdot A \cdot B^{-1}$ $\quad /^{-1}$

$X = B A^{-1} B^{-1} A$

$$A^{-1} = \frac{1}{\det A} \cdot A_{kof}^{T} \qquad \det A = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \qquad A_{kof} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$A_{11} = 1 \quad A_{21} = -1$

$A_{12} = 0 \quad A_{22} = 1$

$$A_{kof}^{T} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{\det B} \cdot B_{kof}^{T}$$

$B_{11} = 1$

$B_{12} = -1$

$B_{21} = 0$

$B_{22} = 1$

$$B_{kof} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad B_{kof}^{T} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\det B = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$B \cdot A^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$B^{-1} \cdot A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$X = B A^{-1} \cdot B^{-1} A = $$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$ traženo rješenje

(#) Riješiti matričnu jednačinu: $AX-2B=3X+A$ gdje je

$$A=\begin{bmatrix}6&1&0\\0&5&2\\0&0&4\end{bmatrix},\quad B=\begin{bmatrix}-1&2&0\\2&3&1\\4&0&3\end{bmatrix}.$$

Rj.

$$AX-2B=3X+A$$

$$AX-3X=2B+A$$

$$\underbrace{(A-3I)}_{M}X=\underbrace{2B+A}_{N}$$

$$MX=N \quad /\cdot M^{-1} \text{ sa lijeve str.}$$

$$M^{-1}MX=M^{-1}\cdot N$$

$$X=M^{-1}\cdot N$$

$$M=A-3I=\begin{bmatrix}6&1&0\\0&5&2\\0&0&4\end{bmatrix}-\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}=\begin{bmatrix}3&1&0\\0&2&2\\0&0&1\end{bmatrix}$$

$$N=2B+A=\begin{bmatrix}-2&4&0\\4&6&2\\8&0&6\end{bmatrix}+\begin{bmatrix}6&1&0\\0&5&2\\0&0&4\end{bmatrix}=\begin{bmatrix}4&5&0\\4&11&4\\8&0&10\end{bmatrix}$$

$$M^{-1}=\frac{1}{\det M}M_{kof}^{T}$$

$$\det M=\begin{vmatrix}3&1&0\\0&2&2\\0&0&1\end{vmatrix}=3\cdot2\cdot1=6$$

$$M_{11}=(-1)^2\begin{vmatrix}2&2\\0&1\end{vmatrix}=2 \qquad M_{21}=(-1)^3\begin{vmatrix}1&0\\0&1\end{vmatrix}=-1$$

$$M_{12}=(-1)^3\begin{vmatrix}0&2\\0&1\end{vmatrix}=0 \qquad M_{22}=(-1)^4\begin{vmatrix}3&0\\0&1\end{vmatrix}=3$$

$$M_{13}=(-1)^4\begin{vmatrix}0&2\\0&0\end{vmatrix}=0 \qquad M_{23}=(1)^5\begin{vmatrix}3&1\\0&0\end{vmatrix}=0$$

$$M_{31}=(-1)^4\begin{vmatrix}1&0\\2&2\end{vmatrix}=2$$

$$M_{32}=(-1)^5\begin{vmatrix}3&0\\0&2\end{vmatrix}=-6$$

$$M_{33}=(-1)^6\begin{vmatrix}3&1\\0&2\end{vmatrix}=6$$

$$M_{kof}=\begin{bmatrix}2&0&0\\-1&3&0\\2&-6&6\end{bmatrix},\qquad M_{kof}^{T}=\begin{bmatrix}2&-1&2\\0&3&-6\\0&0&6\end{bmatrix}$$

$$M^{-1}=\frac{1}{6}\begin{bmatrix}2&-1&2\\0&3&-6\\0&0&6\end{bmatrix},$$

$$X=M^{-1}\cdot N=\frac{1}{6}\begin{bmatrix}2&-1&2\\0&3&-6\\0&0&6\end{bmatrix}\begin{bmatrix}4&5&0\\4&11&4\\8&0&10\end{bmatrix}=\frac{1}{6}\begin{bmatrix}20&-1&16\\-36&33&48\\48&0&60\end{bmatrix}$$

$8-4+16$

$10-11+0$

$0-4+20$

$0+12-48$

$0+33+0$

$12-60$

$$X=\begin{bmatrix}\frac{10}{3}&-\frac{1}{6}&\frac{8}{3}\\-6&\frac{11}{2}&8\\8&0&10\end{bmatrix}$$ traženo rješenje

(#) Riješiti matričnu jednačinu $(XA+B)^{-1}(XC+B)=C$, ako je $A=\begin{bmatrix}-1 & 2 & 3\\ 0 & 2 & -1\\ 0 & 0 & 3\end{bmatrix}$, $B=\begin{bmatrix}2 & -1 & 1\\ 0 & -1 & 1\\ 0 & 0 & 1\end{bmatrix}$ i $C=\begin{bmatrix}-1 & 1 & 2\\ 0 & -1 & 2\\ 0 & 0 & 2\end{bmatrix}$

Rj. $(XA+B)^{-1}(XC+B)=C \quad /(XA+B)$ sa lijeve strane

$$\underbrace{(XA+B)(XA+B)^{-1}}_{I}(XC+B)=(XA+B)\cdot C$$

$$XC+B=XAC+BC$$

$$X=B(C-I)(C-AC)^{-1}$$

$$XC-XAC=BC-B$$

$$C-1=\begin{bmatrix}-2 & 1 & 2\\ 0 & -2 & 2\\ 0 & 0 & 1\end{bmatrix}$$

$X(C-AC)=BC-B \quad /(C-AC)^{-1}$ sa desne strane

$$B(C-1)=\begin{bmatrix}2 & -1 & 1\\ 0 & -1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-2 & 1 & 2\\ 0 & -2 & 2\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-4 & 4 & 3\\ 0 & 2 & -1\\ 0 & 0 & 1\end{bmatrix}$$

$$AC=\begin{bmatrix}-1 & 2 & 3\\ 0 & 2 & -1\\ 0 & 0 & 3\end{bmatrix}\begin{bmatrix}-1 & 1 & 2\\ 0 & -1 & 2\\ 0 & 0 & 2\end{bmatrix}=\begin{bmatrix}1 & -3 & 8\\ 0 & -2 & 2\\ 0 & 0 & 6\end{bmatrix}$$

Označimo sa $D=C-AC=\begin{bmatrix}-2 & 4 & -6\\ 0 & 1 & 0\\ 0 & 0 & -4\end{bmatrix}$

Izračunajmo D^{-1}: $\quad D^{-1}=\dfrac{1}{\det D}D_{kof}^{T}$

$D_{11}=(-1)^2\begin{vmatrix}1 & 0\\ 0 & -4\end{vmatrix}=-4$ $\quad D_{21}=(-1)^3\begin{vmatrix}4 & -6\\ 0 & -4\end{vmatrix}=16$ $\quad D_{31}=(-1)^4\begin{vmatrix}4 & -6\\ 1 & 0\end{vmatrix}=6$

$D_{12}=(-1)^3\begin{vmatrix}0 & 0\\ 0 & -4\end{vmatrix}=0$ $\quad D_{22}=(-1)^4\begin{vmatrix}-2 & -6\\ 0 & -4\end{vmatrix}=8$ $\quad D_{32}=(-1)^5\begin{vmatrix}-2 & -6\\ 0 & 0\end{vmatrix}=0$

$D_{13}=(-1)^4\begin{vmatrix}0 & 1\\ 0 & 0\end{vmatrix}=0$ $\quad D_{23}=(-1)^5\begin{vmatrix}-2 & 4\\ 0 & 0\end{vmatrix}=0$ $\quad D_{33}=(-1)^6\begin{vmatrix}-2 & 4\\ 0 & 1\end{vmatrix}=-2$

$$\det D=\begin{vmatrix}-2 & 4 & -6\\ 0 & 1 & 0\\ 0 & 0 & -4\end{vmatrix}=(-4)\begin{vmatrix}-2 & 4\\ 0 & 1\end{vmatrix}=8 \quad D_{kof}=\begin{bmatrix}-4 & 0 & 0\\ 16 & 8 & 0\\ 6 & 0 & -2\end{bmatrix} \quad D_{kof}^{T}=\begin{bmatrix}-4 & 16 & 6\\ 0 & 8 & 0\\ 0 & 0 & -2\end{bmatrix}$$

$$D^{-1}=\begin{bmatrix}-\frac{1}{2} & 2 & \frac{3}{4}\\ 0 & 1 & 0\\ 0 & 0 & -\frac{1}{4}\end{bmatrix}, \quad X=B(C-1)(C-AC)^{-1}=\frac{1}{8}\begin{bmatrix}-4 & 4 & 3\\ 0 & 2 & -1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-4 & 16 & 6\\ 0 & 8 & 0\\ 0 & 0 & -2\end{bmatrix}$$

$$X=\frac{1}{8}\begin{bmatrix}16 & -32 & -30\\ 0 & 16 & 2\\ 0 & 0 & -2\end{bmatrix}=\begin{bmatrix}2 & -4 & -\frac{15}{4}\\ 0 & 2 & \frac{1}{4}\\ 0 & 0 & -\frac{1}{4}\end{bmatrix}$$ traženo rješenje

(#) Riješiti matričnu jednačinu $XAB = C$, $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$,

$B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 4 & 4 \end{bmatrix}$.

Rj.

$$XAB = C \quad / \cdot (AB)^{-1} \text{ sa desne strane}$$

$$X(AB)(AB)^{-1} = C \cdot (AB)^{-1}$$

$$X = C \cdot (AB)^{-1}$$

$$AB = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 4 & 1 \\ -1 & 2 & 3 \end{bmatrix}$$

$$\det(AB) = \begin{vmatrix} 0 & 2 & 0 \\ 1 & 4 & 1 \\ -1 & 2 & 3 \end{vmatrix} = (-2) \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = (-2)(3+1) = -8$$

AB označimo sa M, nađimo M^{-1}

$$M_{11} = (-1)^2 \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} = 10 \qquad M_{21} = (-1)^3 \begin{vmatrix} 2 & 0 \\ 2 & 3 \end{vmatrix} = -6 \qquad M_{31} = (-1)^4 \begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} = 2$$

$$M_{12} = (-1)^3 \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = -4 \qquad M_{22} = (-1)^4 \begin{vmatrix} 0 & 0 \\ -1 & 3 \end{vmatrix} = 0 \qquad M_{32} = (-1)^5 \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0$$

$$M_{13} = (-1)^4 \begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} = 6 \qquad M_{23} = (-1)^5 \begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix} = -2 \qquad M_{33} = (-1)^6 \begin{vmatrix} 0 & 2 \\ 1 & 4 \end{vmatrix} = -2$$

$$M_{kof} = \begin{bmatrix} 10 & -4 & 6 \\ -6 & 0 & -2 \\ 2 & 0 & -2 \end{bmatrix}, \quad M_{kof}^T = \begin{bmatrix} 10 & -6 & 2 \\ -4 & 0 & 0 \\ 6 & -2 & -2 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{\det M} \cdot M_{kof}^T = \frac{-1}{8} \begin{bmatrix} 10 & -6 & 2 \\ -4 & 0 & 0 \\ 6 & -2 & -2 \end{bmatrix} = \begin{bmatrix} -5/4 & 3/4 & -1/4 \\ 1/2 & 0 & 0 \\ -3/4 & 1/4 & 1/4 \end{bmatrix}$$

$$X = C \cdot (AB^{-1}) = \begin{bmatrix} 0 & 4 & 4 \end{bmatrix} \cdot \left(-\frac{1}{8}\right) \begin{bmatrix} 10 & -6 & 2 \\ -4 & 0 & 0 \\ 6 & -2 & -2 \end{bmatrix} = \left(-\frac{1}{8}\right) \begin{bmatrix} 8 & -8 & -8 \end{bmatrix}$$

$X = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}$ rješenje matrične jednačine

Sistem linearnih jednačina

Sistem od m jednačina sa n nepoznatih zovemo sistem linearnih jednačina

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad \vdots \quad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}$$

Sisteme linearnih jednačina možemo rješiti:

a) Gausovom metodom

b) Kramerovom metodom (metoda determinanti)

c) Matričnom metodom

d) Kroneker-Kapelijevom metodom

① Gausovom metodom rješiti sistem jednačina

$$\begin{array}{ll} x_1 + x_2 - 2x_3 + 4x_4 = -1 & (1) \\ 3x_1 + 2x_2 - x_3 + 3x_4 = 0 & (2) \\ 2x_1 - x_2 + 3x_3 - x_4 = 9 & (3) \\ 5x_1 - 2x_2 + \underline{x_3} - 2x_4 = 9 & (4) \end{array}$$

Rj.

$$\begin{array}{ll} (1)+2\cdot(4): & 11x_1 - 3x_2 = 17 \\ (2)+(4): & 8x_1 + x_4 = 9 \\ (3)-3\cdot(4): & -13x_1 + 5x_2 + 5x_4 = -18 \end{array}$$

$$3x_2 = 11x_1 - 17 \Rightarrow x_2 = \frac{1}{3}(11x_1 - 17)$$

$$x_4 = -8x_1 + 9$$

$$-13x_1 + 5x_2 + 5x_4 = -18$$

$$-13x_1 + \frac{5}{3}(11x_1 - 17) + 5(-8x_1 + 9) = -18$$

$$\underline{-13x_1} + \frac{55}{3}x_1 - \frac{85}{3} \underline{-40x_1} + 45 = -18$$

$$-53x_1 + \frac{55}{3}x_1 = -63 + \frac{85}{3} \quad /\cdot 3$$

$$-159x_1 + 55x_1 = -189 + 85$$

$$-104x_1 = -104$$

$$x_1 = 1$$

$$x_2 = \frac{1}{3}(11x_1 - 17) = \frac{1}{3}(11 - 17) = -2$$

$$x_4 = -8x_1 + 9 = 1$$

$$x_1 + x_2 - 2x_3 + 4x_4 = -1$$

$$-2x_3 = -1 + 2 - 4 - 1$$

$$-2x_3 = -4$$

$$x_3 = 2$$

Rješenje sistema je $x_1 = 1$, $x_2 = -2$, $x_3 = 2$, $x_4 = 1$

② Gausovom metodom riješiti sistem jednačina

$$\begin{array}{l} 2x_1 + 3x_2 - 5x_3 + x_4 - x_5 = 0 \\ x_1 + 2x_2 + 3x_3 + 2x_4 + 2x_5 = 3 \\ 4x_1 + 7x_2 + x_3 + 5x_4 + 3x_5 = 6 \\ 5x_1 + 9x_2 + 4x_3 + 7x_4 + 5x_5 = 9 \end{array}$$

#) Riješiti sistem linearnih jednačina

$$\begin{aligned} 2x_1 - 2x_2 + 2x_3 + 3x_4 &= 1 \\ -2x_1 + x_2 - x_3 - 4x_4 &= 0 \\ 2x_1 - 3x_2 + 3x_3 + 2x_4 &= 2 \\ -x_2 + x_3 - x_4 &= 1 \end{aligned}$$

Rj: Riješimo sistem Gausovom metodom:

$$\begin{aligned} 2x_1 - 2x_2 + 2x_3 + 3x_4 &= 1 \quad (a) \\ -2x_1 + x_2 - x_3 - 4x_4 &= 0 \quad (b) \\ 2x_1 - 3x_2 + 3x_3 + 2x_4 &= 2 \quad (c) \\ -x_2 + x_3 - x_4 &= 1 \quad (d) \end{aligned} \nearrow$$

$$\begin{aligned} (a):\quad 2x_1 - 2x_2 + 2x_3 + 3x_4 &= 1 \\ (b)+(a):\quad -x_2 + x_3 - x_4 &= 1 \\ (c)-(a):\quad -x_2 + x_3 - x_4 &= 1 \\ -x_2 + x_3 - x_4 &= 1 \end{aligned}$$

$$\begin{aligned} 2x_1 - 2x_2 + 2x_3 + 3x_4 &= 1 \\ -x_2 + x_3 - x_4 &= 1 \end{aligned}$$

Imamo dvije linearne jednačine sa četri nepoznate $\Rightarrow$

$\Rightarrow$ dvije promjenjive uzimamo proizvoljno npr. $x_3 = s$, $x_4 = t$

$x_2 = s - t - 1$

$$2x_1 = 1 + 2x_2 - 2x_3 - 3x_4$$

$$2x_1 = 1 + \underline{2s} \underline{\underline{-2t}} - 2 \underline{-2s} \underline{\underline{-3t}}$$

$$2x_1 = -5t - 1$$

$$x_1 = \frac{-5}{2}t - \frac{1}{2}$$

Rješenje sistema linearnih jednačina je

$\left(\frac{-5}{2}t - \frac{1}{2}, s - t - 1, s, t\right)$

Cramerovo pravilo (metoda determinanti)

Rješavamo sistem oblika $A\cdot x=b$ gdje je $A=[a_{ij}]_{n\times n}$, $x=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$

$b=\begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}$. D_k determinanta koja se dobije od D ($D=\det A$) kada se umjesto k-te kolone u D stave slobodni članovi $\begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}$.

a) za $D\neq 0$ sistem ima jedinstveno rješenje $x=\frac{D_x}{D}$, $y=\frac{D_y}{D}$, $z=\frac{D_z}{D}$

b) za $D=0$ i ($D_x\neq 0$ ili $D_y\neq 0$ ili $D_z\neq 0$) sistem nema nijedno rješenje

c) za $D=D_x=D_y=D_z$ ne možemo ništa zaključiti
(sistem može imati ∞ mnogo rješenja ili nemati nijedno rješenje)
(potrebna su dalja ispitivanja)

) Metodom determinanti riješiti sistem jednačina

$$2x-y-z=4$$
$$3x+4y-2z=11$$
$$3x-2y+4z=11.$$

Rj.

$$D=\begin{vmatrix}2&-1&-1\\3&4&-2\\3&-2&4\end{vmatrix}\xlongequal[III_v+I_v\cdot 4]{II_v+I_v\cdot(-2)}\begin{vmatrix}2&-1&-1\\-1&6&0\\11&-6&0\end{vmatrix}=(-1)\begin{vmatrix}-1&6\\11&-6\end{vmatrix}=-(6-66)=60$$

$$D_x=\begin{vmatrix}4&-1&-1\\11&4&-2\\11&-2&4\end{vmatrix}\xlongequal[III_v+I_v\cdot 4]{II_v-I_v\cdot 2}\begin{vmatrix}4&-1&-1\\3&6&0\\27&-6&0\end{vmatrix}=(-1)\begin{vmatrix}3&6\\27&-6\end{vmatrix}=-(-18-162)=180$$

$$D_y=\begin{vmatrix}2&4&-1\\3&11&-2\\3&11&4\end{vmatrix}\xlongequal[II_k+III_k\cdot 4]{I_k+III_k\cdot 2}\begin{vmatrix}0&0&-1\\-1&3&-2\\11&27&4\end{vmatrix}=(-1)\begin{vmatrix}-1&3\\11&27\end{vmatrix}=-(-27-33)=60$$

$$D_z=\begin{vmatrix}2&-1&4\\3&4&11\\3&-2&11\end{vmatrix}\xlongequal[III_v-I_v\cdot 2]{II_v+I_v\cdot 4}\begin{vmatrix}2&-1&4\\11&0&27\\-1&0&3\end{vmatrix}=\begin{vmatrix}11&27\\-1&3\end{vmatrix}=3\begin{vmatrix}11&9\\-1&1\end{vmatrix}=3(11+9)=60$$

$$x=\frac{D_x}{D}=\frac{180}{60}=3;\qquad y=\frac{D_y}{D}=\frac{60}{60}=1;\qquad z=\frac{D_z}{D}=\frac{60}{60}=1$$

Rješenje sistema je $x=3$, $y=1$ i $z=1$

) Metodom determinanti riješiti sistem jednačina:

$$2x+4y-5z=-5$$
$$-x-y+z=0$$
$$2x+y-z=1.$$

Rj. $x=1$, $y=2$, $z=3$

) Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra λ:

$$\begin{aligned}(\lambda-2)x-3y+2z&=1\\ 3x-3y+(\lambda-3)z&=1\\ x-y+2z&=-1.\end{aligned}$$

$$D=\begin{vmatrix}\lambda-2 & -3 & 2\\ 3 & -3 & \lambda-3\\ 1 & -1 & 2\end{vmatrix}\xrightarrow[III_k+II_k\cdot 2]{I_k+II_k}\begin{vmatrix}\lambda-5 & -3 & -4\\ 0 & -3 & \lambda-9\\ 0 & -1 & 0\end{vmatrix}=(\lambda-5)\begin{vmatrix}-3 & \lambda-9\\ -1 & 0\end{vmatrix}=(\lambda-5)(\lambda-9)$$

$$D_x=\begin{vmatrix}1 & -3 & 2\\ 1 & -3 & \lambda-3\\ -1 & -1 & 2\end{vmatrix}\xrightarrow[II_v+III_v]{I_v+III_v}\begin{vmatrix}0 & -4 & 4\\ 0 & -4 & \lambda-1\\ -1 & -1 & 2\end{vmatrix}=(-1)\begin{vmatrix}-4 & 4\\ -4 & \lambda-1\end{vmatrix}=(-1)(-4)\begin{vmatrix}1 & 4\\ 1 & \lambda-1\end{vmatrix}=4(\lambda-5)$$

$$D_y=\begin{vmatrix}\lambda-2 & 1 & 2\\ 3 & 1 & \lambda-3\\ 1 & -1 & 2\end{vmatrix}\xrightarrow[II_v+III_v]{I_v+III_v}\begin{vmatrix}\lambda-1 & 0 & 4\\ 4 & 0 & \lambda-1\\ 1 & -1 & 2\end{vmatrix}=\begin{vmatrix}\lambda-1 & 4\\ 4 & \lambda-1\end{vmatrix}=(\lambda-1)^2-4=(\lambda-1-4)(\lambda-1+4)=(\lambda-5)(\lambda+3)$$

$$D_z=\begin{vmatrix}\lambda-2 & -3 & 1\\ 3 & -3 & 1\\ 1 & -1 & -1\end{vmatrix}\xrightarrow{I_k+II_k}\begin{vmatrix}\lambda-5 & -3 & 1\\ 0 & -3 & 1\\ 0 & -1 & -1\end{vmatrix}=(\lambda-5)\begin{vmatrix}-3 & 1\\ -1 & -1\end{vmatrix}=4(\lambda-5)$$

Diskusija

1° $\lambda\neq 5$ i $\lambda\neq 9$ $(D\neq 0)$ Sistem ima jedinstveno rješenje

$$x=\frac{D_x}{D}=\frac{4(\lambda-5)}{(\lambda-5)(\lambda-9)}=\frac{4}{\lambda-9},\quad y=\frac{D_y}{D}=\frac{\lambda+3}{\lambda-9},\quad z=\frac{D_z}{D}=\frac{4}{\lambda-9}$$

2° $\lambda=9$

$D=0,\ D_x\neq 0 \Rightarrow$ sistem nema rješenja

3° $\lambda=5 \Rightarrow D=D_x=D_y=D_z=0$ na osnovu Cramerovog pravila ne možemo ništa zaključiti. A trebno je uraditi sistem na drugi način.

za $\lambda=5$ sistem postaje

$$\begin{aligned}3x-3y+2z&=1 &(1)\\ 3x-3y+2z&=1 &(2)\\ x-y+2z&=-1 &(3)\end{aligned}$$

$(1)=(2)$

$(2)-(3):\ 2x-2y=2$

$x=y+1$

$x-y+2z=-1$

$y+1-y+2z=-1$

$2z=-2$

$z=-1$

sistem ima beskonačno mnogo rješenja koji su oblika

$(t+1,\ t,\ -1),\ t\in\mathbb{R}$

) Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra λ:

$$\begin{aligned}(\lambda+4)x+y+z&=2\\ x+y+z&=\lambda+5\\ 3x+3y+(\lambda+7)z&=3\end{aligned}$$

Rj:

$D=(\lambda+4)(\lambda+3)$

$D_x=-(\lambda+4)(\lambda+3)$

$D_y=(\lambda+3)(\lambda+4)(\lambda+8)$

$D_z=-3(\lambda+3)(\lambda+4)$

$t\in\mathbb{R}$

$(t, 5-t, -3)$

$(-1, 2-s, s)$

$s\in\mathbb{R}$

(#) Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra λ

$$\begin{aligned} x+y+z&=4\\ x+\lambda y+z&=3\\ x+2\lambda y+z&=4 \end{aligned} \quad .$$

Rj. Sistem rješavamo Cramerovom metodom

$$D=\begin{vmatrix}1&1&1\\1&\lambda&1\\1&2\lambda&1\end{vmatrix} \overset{II_v-III_v}{=\!=} \begin{vmatrix}1&1&1\\0&-\lambda&0\\1&2\lambda&1\end{vmatrix} = -\lambda\begin{vmatrix}1&1\\1&1\end{vmatrix}=0$$

$$D_x=\begin{vmatrix}4&1&1\\3&\lambda&1\\4&2\lambda&1\end{vmatrix} \underset{III_v-II_v}{\overset{I_v-II_v}{=\!=}} \begin{vmatrix}1&1-\lambda&0\\3&\lambda&1\\1&\lambda&0\end{vmatrix} = -\begin{vmatrix}1&1-\lambda\\1&\lambda\end{vmatrix} = -(\lambda-(1-\lambda)) = 1-\lambda-\lambda = 1-2\lambda$$

$$D_y=\begin{vmatrix}1&4&1\\1&3&1\\1&4&1\end{vmatrix} \overset{III_k-I_k}{=\!=} \begin{vmatrix}1&4&0\\1&3&0\\1&4&0\end{vmatrix} = 0$$

$$D_z=\begin{vmatrix}1&1&4\\1&\lambda&3\\1&2\lambda&4\end{vmatrix} \underset{III_v-II_v}{\overset{I_v-II_v}{=\!=}} \begin{vmatrix}0&1-\lambda&1\\1&\lambda&3\\0&\lambda&1\end{vmatrix} = -\begin{vmatrix}1-\lambda&1\\\lambda&1\end{vmatrix} = -(1-\lambda-\lambda) = 2\lambda-1$$

Kako je $D=0$ to sistem može da ima beskonačno mnogo rješenja ili da nema rješenja.

1° $\lambda=\frac{1}{2}$

$D=0,\ D_x=0,\ D_y=0,\ D_z=0$

Sistem ćemo riješiti Gauss-ovom metodom

$$\begin{aligned} x+y+z&=4\\ x+\tfrac{1}{2}y+z&=3 \quad /\cdot 2\\ x+y+z&=4 \end{aligned}$$

$$\begin{aligned} x+y+z&=4 \quad (1)\\ 2x+y+2z&=6 \quad (2) \end{aligned}$$

$(2)-(1):\ x+z=2$

$x=2-z$

$$\begin{aligned} x+y+z&=4\\ 2-z+y+z&=4\\ y&=2 \end{aligned}$$

Za $\lambda=\frac{1}{2}$ sistem ima ∞ mnogo rješenja koja su oblika $(2-t, 2, t)$ gdje je $t\in\mathbb{R}$.

2° $\lambda\neq\frac{1}{2}$

$D=0,\ D_x\neq 0 \Rightarrow$ sistem za $\lambda\neq\frac{1}{2}$ nema rješenja

(#) Odrediti vrijednost parametra k tako da sistem

$$\begin{aligned} 8z - 3x - 6y &= kx \\ 2x + y + 4z &= ky \\ 4x + 3y + z &= kz \end{aligned}$$

ima beskonačno mnogo rješenja. Zatim naći ta rješenja za najveću dobijenu vrijednost parametra k.

Rj. Nepoznate sa desne strane prebacimo na lijevu i grupirajmo vrijednosti uz x, y i z.

$$\begin{aligned} (-3-k)x - 6y + 8z &= 0 \\ 2x + (1-k)y + 4z &= 0 \\ 4x + 3y + (1-k)z &= 0 \end{aligned}$$

$$\begin{vmatrix} -3-k & -6 & 8 \\ 2 & 1-k & 4 \\ 4 & 3 & 1-k \end{vmatrix} = 0$$

$I_k + III_k$: $$\begin{vmatrix} 5-k & -6 & 8 \\ 6 & 1-k & 4 \\ 5-k & 3 & 1-k \end{vmatrix} = 0$$

$I_v - III_v$: $$\begin{vmatrix} 0 & -9 & 7+k \\ 6 & 1-k & 4 \\ 5-k & 3 & 1-k \end{vmatrix} = 0$$

Ovo je homogeni sistem linearnih jednačina. Trivijalno rješenje je (0,0,0). Sistem ima beskonačno mnogo rješenja ako je $D=0$.

$$-9+9k \qquad -21-3k \qquad 7-6k-k^2 \qquad 7-7k+k-k^2$$

$$6[(-9)(1-k) - 3(7+k)] + (5-k)[(-9)\cdot 4 - (7+k)(1-k)] = 0$$

$$(-6)(6k-30) + (5-k)(\overbrace{-36-7}^{-43}+6k+k^2) = 0$$

$$\underline{\underline{-36k}} + 180 + (-215) + \underline{\underline{30k}} + \underline{5k^2} + \underline{\underline{43k}} \underline{-6k^2} - k^3 = 0$$

$$-k^3 - k^2 + 37k - 35 = 0 \quad / \cdot (-1)$$

$$k^3 + k^2 - 37k + 35 = 0$$

$$k^3 - k^2 + 2k^2 - 2k - 35k + 35 = 0$$

$$k^2(1-k) + 2k(k-1) - 35(k-1) = 0$$

$$(k-1)(k^2+2k-35) = 0$$

$$(k-1)(k+7)(k-5) = 0$$

$$k_1 = 1, \; k_2 = -7, \; k_3 = 5$$

Za $k=5$ imamo.

$$\begin{aligned} 8x + 6y - 8z &= 0 \quad \ldots(1) \\ 2x - 4y + 4z &= 0 \quad \ldots(2) \\ 4x + 3y - 4z &= 0 \quad \ldots(3) \end{aligned}$$

(2) + (3): $6x - y = 0$

$\Rightarrow y = 6x$

(2) $\rightarrow$ $2x - 24x + 4z = 0$

$\therefore 4z = 22x$

$z = \frac{11x}{2}$

(1) = (3) jer se (3) dobija dijeljenjem (1) sa 2.

Za $k=5$ sistem ima rješenje $(t, 6t, \frac{11t}{2})$ gdje je $t \in \mathbb{R}$ proizvoljno.

(#) Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra λ:

$$\begin{aligned} x - y - \lambda z &= 1 \\ (\lambda+1)y + (\lambda-1)z &= 0 \\ (\lambda+1)x - (\lambda+1)z &= 1 \end{aligned}$$

Rj.

$$D = \begin{vmatrix} 1 & -1 & -\lambda \\ 0 & \lambda+1 & \lambda-1 \\ \lambda+1 & 0 & -(\lambda+1) \end{vmatrix} \overset{III_k + I_k}{=\!=} \begin{vmatrix} 1 & -1 & 1-\lambda \\ 0 & \lambda+1 & \lambda-1 \\ \lambda+1 & 0 & 0 \end{vmatrix} = (\lambda+1)\begin{vmatrix} -1 & -(\lambda-1) \\ \lambda+1 & \lambda-1 \end{vmatrix} =$$

$$= (\lambda+1)(\lambda-1)\begin{vmatrix} -1 & -1 \\ \lambda+1 & 1 \end{vmatrix} = \lambda(\lambda-1)(\lambda+1)$$

$$D_x = \begin{vmatrix} 1 & -1 & -\lambda \\ 0 & \lambda+1 & \lambda-1 \\ 1 & 0 & -(\lambda+1) \end{vmatrix} \overset{III_v - I_v}{=\!=} \begin{vmatrix} 1 & -1 & -\lambda \\ 0 & \lambda+1 & \lambda-1 \\ 0 & 1 & -1 \end{vmatrix} \overset{-1+\lambda+1}{} = \begin{vmatrix} \lambda+1 & \lambda-1 \\ 1 & -1 \end{vmatrix} = \lambda-1-\lambda+1$$

$$= -2\lambda$$

$$D_y = \begin{vmatrix} 1 & 1 & -\lambda \\ 0 & 0 & \lambda-1 \\ \lambda+1 & 1 & -(\lambda+1) \end{vmatrix} = -(\lambda-1)\begin{vmatrix} 1 & 1 \\ \lambda+1 & 1 \end{vmatrix} = -(\lambda-1)(1-\lambda-1) = \lambda(\lambda-1)$$

$$D_z = \begin{vmatrix} 1 & -1 & 1 \\ 0 & \lambda+1 & 0 \\ \lambda+1 & 0 & 1 \end{vmatrix} = (\lambda+1)\begin{vmatrix} 1 & 1 \\ \lambda+1 & 1 \end{vmatrix} = -\lambda(\lambda+1)$$

$D=0$ akko $\lambda=0$ ili $\lambda=1$ ili $\lambda=-1$

Diskusija

1° $\lambda \neq 0$ i $\lambda \neq 1$ i $\lambda \neq -1$ sistem ima jedinstveno rješenje

$$x = \frac{D_x}{D} = \frac{-2\lambda}{\lambda(\lambda-1)(\lambda+1)} = \frac{-2}{(\lambda-1)(\lambda+1)}, \quad y = \frac{D_y}{D} = \frac{1}{\lambda+1}, \quad z = \frac{D_z}{D} = \frac{-1}{\lambda+1}$$

2° $\lambda=1$, $D=0$, $D_x \neq 0$ $\Rightarrow$ sistem nema rješenja

3° $\lambda=-1$, $D=0$, $D_x \neq 0$ $\Rightarrow$ sistem nema rješenja

4° $\lambda=0$, $D=D_x=D_y=D_z=0$ iz ovoga ne možemo ništa zaključiti

Za $\lambda=0$ sistem postaje

$$\begin{aligned} x - y \quad &= 1 \quad (1) \\ y - z &= 0 \quad (2) \\ x \quad - z &= 1 \quad (3) \end{aligned}$$

(1): $x - y = 1$

(2)−(3): $\underline{-x + y = -1}$

$x = y+1$

$x - z = 1$

$-z = -(y+1)+1$

$-z = -y$

$z = y$

Sistem ima ∞ mnogo rješenja $(t+1, t, t)$, $t \in \mathbb{R}$

#) Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra a:

$$x+y-z=0$$
$$x-y+az=1$$
$$-x-3y+(a+2)z=a^2 .$$

Rj.

$$D=\begin{vmatrix}1 & 1 & -1\\ 1 & -1 & a\\ -1 & -3 & a+2\end{vmatrix} \overset{I_k+III_k}{\underset{II_k+III_k}{=\!=}} \begin{vmatrix}0 & 0 & -1\\ a+1 & a-1 & a\\ a+1 & a-1 & a+2\end{vmatrix} = (-1)\begin{vmatrix}a+1 & a-1\\ a+1 & a-1\end{vmatrix} = 0 .$$

$$D_x=\begin{vmatrix}0 & 1 & -1\\ 1 & -1 & a\\ a^2 & -3 & a+2\end{vmatrix} \overset{II_k+III_k}{=\!=} \begin{vmatrix}0 & 0 & -1\\ 1 & a-1 & a\\ a^2 & a-1 & a+2\end{vmatrix} = (-1)\begin{vmatrix}1 & a-1\\ a^2 & a-1\end{vmatrix} = (-1)(a-1)\begin{vmatrix}1 & 1\\ a^2 & 1\end{vmatrix}$$
$$=(-1)(a-1)(1-a^2)=(a-1)(a^2-1)$$
$$=(a-1)^2(a+1)$$

$$D_y=\begin{vmatrix}1 & 0 & -1\\ 1 & 1 & a\\ -1 & a^2 & a+2\end{vmatrix} \overset{I_k+III_k}{=\!=} \begin{vmatrix}0 & 0 & -1\\ a+1 & 1 & a\\ a+1 & a^2 & a+2\end{vmatrix} = (-1)\begin{vmatrix}a+1 & 1\\ a+1 & a^2\end{vmatrix} = (-1)(a+1)\begin{vmatrix}1 & 1\\ 1 & a^2\end{vmatrix} = (-1)(a+1)(a^2-1)$$
$$=(-1)(a-1)(a+1)^2$$

$$D_z=\begin{vmatrix}1 & 1 & 0\\ 1 & -1 & 1\\ -1 & -3 & a^2\end{vmatrix} \overset{I_k-II_k}{=\!=} \begin{vmatrix}0 & 1 & 0\\ 2 & -1 & 1\\ 2 & -3 & a^2\end{vmatrix} = (-1)\begin{vmatrix}2 & 1\\ 2 & a^2\end{vmatrix} = (-1)(2a^2-2) = (-2)(a+1)(a-1)$$

Diskusija

$D=0 \quad \forall a\in \mathbb{R}$

1° $a\neq 1$; $a\neq -1$

$D=0$; $D_x\neq 0$ sistem nema rješenja

2° $a=1$

$D=D_x=D_y=D_z=0$, . sistem postaje

$$\begin{aligned} x+y-z&=0 && (1)\\ x-y+z&=1 && (2)\\ -x-3y+3z&=1 && (3)\end{aligned}$$

$(1)+(3)$: $-2y+2z=1$
$(2)+(3)$: $-4y+4z=2$

$2z=2y+1$

$z=y+\frac{1}{2}$

$x=z-y$

$x=\frac{1}{2}$

Sistem ima ∞ mnogo rješenja oblika $(\frac{1}{2}, t, t+\frac{1}{2})$ gdje je $t\in \mathbb{R}$.

3° $a=-1$

$D=D_x=D_y=D_z=0$, sistem postaje

$$\begin{aligned} x+y-z&=0 && I\\ x-y-z&=1 && II\\ -x-3y+z&=1 && III\end{aligned}$$

$(I)+(III)$: $-2y=1$
$(II)+(III)$: $-4y=2$

$y=-\frac{1}{2}$

$(I)+(II)$: $2x-2z=1$
$(III)-3\cdot(II)$: $-4x+4z=2$

$2x=2z+1$

$x=z+\frac{1}{2}$

Sistem ima ∞ mnogo rješenja oblika $(t+\frac{1}{2}, -\frac{1}{2}, t)$, $t\in \mathbb{Z}$

(#) Diskutovati rješenja sistema u zavisnosti od parametra λ:

$$\begin{aligned} 2x - \lambda y + 2z &= 1 \\ x + y + 2z &= 0 \\ -x + (-\lambda-3)y - 4z &= \lambda . \end{aligned}$$

Rj: Sistem ćemo riješiti Cramerovim pravilima.

$$D = \begin{vmatrix} 2 & -\lambda & 2 \\ 1 & 1 & 2 \\ -1 & -\lambda-3 & -4 \end{vmatrix} \underset{III_k - II_k \cdot 2}{\overset{I_k - II_k}{=\!=}} \begin{vmatrix} 2+\lambda & -\lambda & 2\lambda+2 \\ 0 & 1 & 0 \\ \lambda+2 & -\lambda-3 & 2\lambda+2 \end{vmatrix} = \begin{vmatrix} \lambda+2 & 2\lambda+2 \\ \lambda+2 & 2\lambda+2 \end{vmatrix} = (\lambda+2) \begin{vmatrix} 1 & 2\lambda+2 \\ 1 & 2\lambda+2 \end{vmatrix} = 0$$

$$D_x = \begin{vmatrix} 1 & -\lambda & 2 \\ 0 & 1 & 2 \\ \lambda & -\lambda-3 & -4 \end{vmatrix} \overset{III_k - II_k \cdot 2}{=\!=} \begin{vmatrix} 1 & -\lambda & 2\lambda+2 \\ 0 & 1 & 0 \\ \lambda & -\lambda-3 & 2\lambda+2 \end{vmatrix} = \begin{vmatrix} 1 & 2\lambda+2 \\ \lambda & 2\lambda+2 \end{vmatrix} = (2\lambda+2) \begin{vmatrix} 1 & 1 \\ \lambda & 1 \end{vmatrix}$$

$$= (2\lambda+2)(1-\lambda) = 2(\lambda+1)(1-\lambda)$$

$$D_y = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ -1 & \lambda & -4 \end{vmatrix} \overset{III_k - I_k \cdot 2}{=\!=} \begin{vmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ -1 & \lambda & -2 \end{vmatrix} = (-1) \begin{vmatrix} 1 & -2 \\ \lambda & -2 \end{vmatrix} = (-1)(-2) \begin{vmatrix} 1 & 1 \\ \lambda & 1 \end{vmatrix} = 2(1-\lambda)$$

$$D_z = \begin{vmatrix} 2 & -\lambda & 1 \\ 1 & 1 & 0 \\ -1 & -\lambda-3 & \lambda \end{vmatrix} \overset{I_k - II_k}{=\!=} \begin{vmatrix} 2+\lambda & -\lambda & 1 \\ 0 & 1 & 0 \\ \lambda+2 & -\lambda-3 & \lambda \end{vmatrix} = \begin{vmatrix} \lambda+2 & 1 \\ \lambda+2 & \lambda \end{vmatrix} = (\lambda+2) \begin{vmatrix} 1 & 1 \\ 1 & \lambda \end{vmatrix} =$$

$$(\lambda+2)(\lambda-1)$$

Diskusija:

$D=0$, $D_x = 2(1+\lambda)(1-\lambda)$, $D_y = 2(1-\lambda)$, $D_z = (\lambda+2)(\lambda-1)$

1° $\lambda \neq -1$; $\lambda \neq 1$; $\lambda \neq -2$

imamo $D=0$; $D_x \neq 0$ sistem nema rješenja

2° $\lambda = -2$ imamo $D=0$; $D_x \neq 0$ sistem nema rješenja

3° $\lambda = -1$ imamo $D=0$, $D_x = 0$, $D_y \neq 0$ sistem nema rješenja

4° $\lambda = 1$ imamo $D = D_x = D_y = D_z = 0$ sistem je potrebno ispitati na drugi način.

Za $\lambda = 1$ sistem postaje

$$\begin{aligned} 2x - y + 2z &= 1 \quad / \cdot 4 \\ x + y + 2z &= 0 \quad / \cdot 4 \\ -x - 4y - 4z &= 1 \end{aligned}$$

$$\begin{aligned} 8x - 4y + 8z &= 4 \quad (1) \\ 4x + 4y + 8z &= 0 \quad (2) \\ -x - 4y - 4z &= 1 \quad (3) \end{aligned}$$

$(1)+(2)$: $12x + 16z = 4$

$(3)+(2)$: $3x + 4z = 1$

$3x = 1 - 4z$

$x = \frac{1-4z}{3}$

$y = -x - 2z$

$y = \frac{4z-1}{3} - \frac{6z}{3}$

Sistem ima ∞ mnogo rješenja, oblika $\left(\frac{1-4t}{3}, \frac{-2t-1}{3}, t\right)$ $t \in \mathbb{R}$

(#) Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra

$$\begin{aligned} x+y+bz &= 1-b \\ x-by-z &= 2 \\ bx-y+z &= 2b \end{aligned}$$

Rj. Rješavamo sistem Cramerovom metodom

$$D=\begin{vmatrix} 1 & 1 & b \\ 1 & -b & -1 \\ b & -1 & 1 \end{vmatrix} \overset{I_k+III_k}{=} \begin{vmatrix} b+1 & 1 & b \\ 0 & -b & -1 \\ b+1 & -1 & 1 \end{vmatrix} = (b+1)\begin{vmatrix} 1 & 1 & b \\ 0 & -b & -1 \\ 1 & -1 & 1 \end{vmatrix} \overset{I_v-III_v}{=}$$

$$=(b+1)\begin{vmatrix} 0 & 2 & b-1 \\ 0 & -b & -1 \\ 1 & -1 & 1 \end{vmatrix} = (b+1)\begin{vmatrix} 2 & b-1 \\ -b & -1 \end{vmatrix} = (b+1)\left[\overbrace{-2+(b^2-b)}^{b^2-b-2}\right] =$$

$$=(b+1)(b+1)(b-2)$$

$$D_x=\begin{vmatrix} 1-b & 1 & b \\ 2 & -b & -1 \\ 2b & -1 & 1 \end{vmatrix} \overset{I_v+III_v}{=} \begin{vmatrix} b+1 & 0 & b+1 \\ 2 & -b & -1 \\ 2b & -1 & 1 \end{vmatrix} = (b+1)\begin{vmatrix} 1 & 0 & 1 \\ 2 & -b & -1 \\ 2b & -1 & 1 \end{vmatrix} =$$

$$\overset{I_k-III_k}{=} (b+1)\begin{vmatrix} 0 & 0 & 1 \\ 3 & -b & -1 \\ 2b-1 & -1 & 1 \end{vmatrix} = (b+1)\begin{vmatrix} 3 & -b \\ 2b-1 & -1 \end{vmatrix} = (b+1)\overbrace{(-3+2b^2-b)}^{2b^2-b-3} =$$

$D=1+24=25$

$b_{1,2}=\frac{1\pm5}{4}$

$$=(b+1)\cdot 2\left(b-\frac{3}{2}\right)(b+1)$$

$$=(b+1)(2b-3)(b+1)$$

$$D_y=\begin{vmatrix} 1 & 1-b & b \\ 1 & 2 & -1 \\ b & 2b & 1 \end{vmatrix} \overset{I_k+III_k}{=} \begin{vmatrix} b+1 & 1-b & b \\ 0 & 2 & -1 \\ b+1 & 2b & 1 \end{vmatrix} = (b+1)\begin{vmatrix} 1 & 1-b & b \\ 0 & 2 & -1 \\ 1 & 2b & 1 \end{vmatrix} \overset{III_v-I_v}{=}$$

$$=(b+1)\begin{vmatrix} 1 & 1-b & b \\ 0 & 2 & -1 \\ 0 & 3b-1 & 1-b \end{vmatrix} = (b+1)\begin{vmatrix} 2 & -1 \\ 3b-1 & 1-b \end{vmatrix} = (b+1)(2-2b+3b-1) =$$

$$=(b+1)(b+1)$$

$$D_z=\begin{vmatrix} 1 & 1 & 1-b \\ 1 & -b & 2 \\ b & -1 & 2b \end{vmatrix} \overset{I_v+III_v}{=} \begin{vmatrix} b+1 & 0 & b+1 \\ 1 & -b & 2 \\ b & -1 & 2b \end{vmatrix} = (b+1)\begin{vmatrix} 1 & 0 & 1 \\ 1 & -b & 2 \\ b & -1 & 2b \end{vmatrix} \overset{I_k-III_k}{=}$$

$$=(b+1)\begin{vmatrix} 0 & 0 & 1 \\ -1 & -b & 2 \\ -b & -1 & 2b \end{vmatrix} = (b+1)\begin{vmatrix} -1 & -b \\ -b & -1 \end{vmatrix} = (b+1)(1-b^2) = -(b+1)(b^2-1)$$

$$=-(b+1)(b-1)(b+1)$$

Diskusija: a) $D\neq0$ tj. $b\neq-1$ i $b\neq2$

sistem ima jedinstveno rješenje $x=\frac{D_x}{D}=\frac{(2b-3)(b+1)^2}{(b+1)^2(b-2)}=\frac{2b-3}{b-2}$

$y=\frac{D_y}{D}=\frac{(b+1)^2}{(b+1)^2(b-2)}=\frac{1}{b-2}$; $z=\frac{D_z}{D}=\frac{-(b-1)(b+1)^2}{(b-2)(b+1)^2}=-\frac{b-1}{b-2}$

b) $b=-1 \Rightarrow D=D_x=D_y=D_z=0$ sistem trebamo rješiti na drugi način

Za $b=-1$ sistem postaje
$$\begin{aligned} x+y-z&=2 \\ x+y-z&=2 \\ -x-y+z&=-2 \quad /\cdot(-1) \end{aligned}$$

Sve tri jednačine su iste $\Rightarrow$ Sistem ima ∞ mnogo rješenja. Ako uzmemo $x=t$, $y=s$ rješenja sistema su
$(t, s, t+s-2)$ ↖ dvije promjenjive uzimamo proizvoljno

c) $b=2 \Rightarrow D=0,\ D_x=9\neq 0 \Rightarrow$
Sistem za $b=2$ nema rješenja

Kroneker-Kapelijeva metoda

Neka je dat sistem linearnih jednačina $Ax=b$, gdje su

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn}\end{bmatrix}_{m\times n}, \quad x=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} \quad i \quad b=\begin{bmatrix}b_1\\b_2\\\vdots\\b_m\end{bmatrix}.$$

Matricu $\overline{A}=[A\,|\,b]$ zovemo proširena matrica.

Teorema (Kroneker-Kapeli):

Sistem ima jedinstveno rješenje ako i samo ako je $\text{rang}\,A=\text{rang}\,\overline{A}=n$ (n broj nepoznatih).

Ako je $\text{rang}\,A=\text{rang}\,\overline{A}<n$ tada sistem ima ∞ mnogo rješenja. ($n-\text{rang}\,A$ nepoznatih uzima se proizvoljno)

Ako je $\text{rang}\,A<\text{rang}\,\overline{A}$ tada sistem nema rješenja.

① Kroneker-Kapelijevom metodom riješiti sistem jednačina

$$\begin{aligned} 2x+4y-5z&=-5\\ -x-y+z&=0\\ 2x+y-z&=1.\end{aligned}$$

Rj.

$$\overline{A}=[A\,|\,b]=\left[\begin{array}{ccc|c} 2 & 4 & -5 & -5\\ -1 & -1 & 1 & 0\\ 2 & 1 & -1 & 1\end{array}\right] \overset{I_v\leftrightarrow II_v}{\sim} \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0\\ 2 & 4 & -5 & -5\\ 2 & 1 & -1 & 1\end{array}\right] \underset{III_v+I_v\cdot 2}{\overset{II_v+I_v\cdot 2}{\sim}} \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0\\ 0 & 2 & -3 & -5\\ 0 & -1 & 1 & 1\end{array}\right]$$

$$\overset{II_v\leftrightarrow III_v}{\sim} \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0\\ 0 & -1 & 1 & 1\\ 0 & 2 & -3 & -5\end{array}\right] \overset{III_v+II_v\cdot 2}{\sim} \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0\\ 0 & -1 & 1 & 1\\ 0 & 0 & -1 & -3\end{array}\right]$$

$\text{rang}\,A=\text{rang}\,\overline{A}=3$

sistem ima jedinstveno rješenje

$$\begin{aligned} -x-y+z&=0\\ -y+z&=1\\ -z&=-3\end{aligned}$$

$z=3$

$$\begin{aligned} -x-y&=-3\\ -y&=-2\end{aligned}$$

$y=2$

$-x-2=-3$

$x=1$

Rješenje sistema je uređena trojka $(1,2,3)$.

② Kroneker-Kapelijevom metodom riješiti sistem jednačina

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 3x_1 + x_2 - x_3 &= 3 \\ 2x_1 + x_2 \phantom{{}+x_3} &= 2. \end{aligned}$$

Rj. $\bar{A} = [A \mid b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & 3 \\ 2 & 1 & 0 & 2 \end{array}\right] \underset{III_v - I_v \cdot 2}{\overset{II_v - I_v \cdot 3}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -4 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \overset{II_v \leftrightarrow III_v}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right]$

$\overset{III_v - II_v \cdot 2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\text{rang}\, A = \text{rang}\, \bar{A} = 2 < 3$

sistem ima ∞ mnogo rješenja

3-2 nepoznatih uzimamo proizvoljno

$x_3 = t$ $\qquad -x_2 - 2t = 0 \qquad x_1 - 2t + t = 1$

$-x_2 - 2x_3 = 0 \qquad x_2 = -2t \qquad x_1 = t + 1$

$x_1 + x_2 + x_3 = 1$

Sistem ima beskonačno mnogo rješenja oblika $(t+1, -2t, t)$ gdje je $t \in \mathbb{R}$.

③ Kroneker-Kapelijevom metodom riješiti sistem jednačina

$$\begin{aligned} x + 2y + 3z &= 1 \\ 2x + 4y + 6z &= 2 \\ 3x + 6y + 9z &= 5. \end{aligned}$$

Rj. $\bar{A} = [A \mid b] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 3 & 6 & 9 & 5 \end{array}\right] \underset{III_v - I_v \cdot 3}{\overset{II_v - I_v \cdot 2}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$

$\text{rang}\, A = 1$, $\text{rang}\, \bar{A} = 2$, $\text{rang}\, A < \text{rang}\, \bar{A}$

sistem nema rješenja.

④ Kroneker-Kapelijevom metodom diskutovati rješenja sistema za razne vrijednosti parametra λ

$$\begin{aligned} \lambda x + y + z &= 1 \\ x + \lambda y + z &= 2 \\ x + y + \lambda z &= -3. \end{aligned}$$

Rj. za $\lambda \in (-\infty, -2) \cup (-2, 1) \cup (1, +\infty)$ sistem ima jedinstveno rješenje $\left(\frac{1}{\lambda-1}, \frac{2}{\lambda-1}, \frac{-3}{\lambda-1}\right)$

za $\lambda = -2$ sistem ima ∞ mnogo rješenja $\left(\frac{3t-4}{3}, \frac{3t-5}{3}, t\right)$, $t \in \mathbb{R}$

za $\lambda = 1$ sistem nema rješenja

(#) Riješiti sistem jednačina za razne vrijednosti parametra $\lambda \in \mathbb{R}$:

$$\begin{aligned} 2x_1 - x_2 + 3x_3 - 7x_4 &= 15 \\ 6x_1 - 3x_2 + x_3 - 4x_4 &= 7 \\ 4x_1 - 2x_2 + 14x_3 - 31x_4 &= \lambda \end{aligned}$$

Rj.

Rješimo sistem Kroneker-Kapelijevom metodom:

$$\overline{C} = [C \mid b] = \left[\begin{array}{cccc|c} 2 & -1 & 3 & -7 & 15 \\ 6 & -3 & 1 & -4 & 7 \\ 4 & -2 & 14 & -31 & \lambda \end{array}\right] \underset{III_v - I_v \cdot 2}{\overset{II_v - I_v \cdot 3}{\sim}} \left[\begin{array}{cccc|c} 2 & -1 & 3 & -7 & 15 \\ 0 & 0 & -8 & 17 & -38 \\ 0 & 0 & 8 & -17 & \lambda - 30 \end{array}\right]$$

$$\overset{III_v + II_v}{\sim} \left[\begin{array}{cccc|c} 2 & -1 & 3 & -7 & 15 \\ 0 & 0 & -8 & 17 & -38 \\ 0 & 0 & 0 & 0 & \lambda - 68 \end{array}\right]$$

1° $\lambda - 68 \neq 0$
$\lambda \neq 68$

$\text{rang}\, C = 2$
$\text{rang}\, \overline{C} = 3$

$\text{rang}\, C < \text{rang}\, \overline{C}$

Prema Kroneker-Kapelijevoj teoremi sistem nema rješenje

2° $\lambda - 68 = 0$
$\lambda = 68$

$\text{rang}\, C = \text{rang}\, \overline{C} = 2 < 4$ (broj nepoznatih)

Prema Kroneker-Kapelijevoj teoremi dvije promjenjive uzimamo proizvoljno, npr. $x_4 = t$, $x_1 = s$

$$\begin{aligned} 2x_1 - x_2 + 3x_3 - 7x_4 &= 15 \\ -8x_3 + 17x_4 &= -38 \end{aligned}$$

$x_4 = t$

$$-8x_3 + 17t = -38$$
$$-8x_3 = -17t - 38$$
$$x_3 = \frac{17}{8}t + \frac{38}{8} = \frac{17}{8}t + \frac{19}{4}$$

$x_1 = s$

$$2s - x_2 + 3\left(\frac{17}{8}t + \frac{38}{8}\right) - 7t = 15$$
$$x_2 = \frac{51t}{8} + \frac{114}{8} + 2s - 7t - 15$$
$$x_2 = -\frac{5}{8}t - \frac{6}{8} + 2s$$
$$x_2 = 2s - \frac{5}{8}t - \frac{3}{4}$$

Za $\lambda = 68$ rješenje sistema je

$$\left(s,\ 2s - \frac{5}{8}t - \frac{3}{4},\ \frac{17}{8}t + \frac{19}{4},\ t\right),\quad t, s \in \mathbb{R}$$

(#) Riješiti sistem jednačina za razne vrijednosti parametra $\lambda \in \mathbb{R}$:

$$\begin{aligned} 8x_1 + 12x_2 + 7x_3 + \lambda x_4 &= 9 \\ 6x_1 + 9x_2 + 5x_3 + 6x_4 &= 7 \\ 4x_1 + 6x_2 + 3x_3 + 4x_4 &= 5 \\ 2x_1 + 3x_2 + 2x_3 + 2x_4 &= 2. \end{aligned}$$

Rj. Sistem ćemo riješiti Kroneker-Kapelijevom metodom:

$$\overline{B} = [B \mid b] = \left[\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 8 & 12 & 7 & \lambda & 9 \\ 6 & 9 & 5 & 6 & 7 \\ 4 & 6 & 3 & 4 & 5 \\ 2 & 3 & 2 & 2 & 2 \end{array}\right] \overset{I_v \leftrightarrow IV_v}{\sim} \left[\begin{array}{cccc|c} 2 & 3 & 2 & 2 & 2 \\ 6 & 9 & 5 & 6 & 7 \\ 4 & 6 & 3 & 4 & 5 \\ 8 & 12 & 7 & \lambda & 9 \end{array}\right] \overset{\substack{II_v - I_v \cdot 3 \\ III_v - I_v \cdot 2 \\ IV_v - I_v \cdot 4}}{\sim}$$

$$\sim \left[\begin{array}{cccc|c} 2 & 3 & 2 & 2 & 2 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & \lambda-8 & 1 \end{array}\right] \overset{\substack{III_v - II_v \\ IV_v - II_v}}{\sim} \left[\begin{array}{cccc|c} 2 & 3 & 2 & 2 & 2 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda-8 & 0 \end{array}\right]$$

1° za $\lambda = 8$ imamo $\operatorname{rang} B = \operatorname{rang} \overline{B} = 2 < 4$ pa prema Kroneker-Kapelijevoj teoremi sistem ima ∞ mnogo rješenja. Dvije promjenjive uzimamo proizvoljno npr. $x_1 = t$, $x_4 = s$

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 + 2x_4 &= 2 \\ -x_3 + 0x_4 &= 1 \end{aligned} \qquad \begin{aligned} x_3 &= -1 \\ 2t + 3x_2 - 2 + 2s &= 2 \end{aligned} \qquad \begin{aligned} 3x_2 &= 4 - 2t - 2s \\ x_2 &= \frac{2}{3}(2 - t - s) \end{aligned}$$

Rješenje sistema (za $\lambda = 8$) je $(t, \frac{2}{3}(2-t-s), -1, s)$ gdje su $s, t \in \mathbb{R}$.

2° za $\lambda \neq 8$ imamo $\operatorname{rang} B = \operatorname{rang} \overline{B} = 3 < 4$ pa prema Kroneker-Kapelijevoj teoremi sistem ima ∞ mnogo rješenja. Jednu promjenjivu uzimamo proizvoljno npr. $x_2 = t$.

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 + 2x_4 &= 2 \\ -x_3 &= 1 \\ (\lambda - 8)x_4 &= 0 \end{aligned} \qquad \begin{aligned} x_4 &= 0 \\ x_3 &= -1 \\ 2x_1 + 3t - 2 &= 2 \end{aligned} \qquad \begin{aligned} 2x_1 &= 4 - 3t \\ x_1 &= 2 - \frac{3}{2}t \end{aligned}$$

Rješenje sistema (za $\lambda \neq 8$) je $(2 - \frac{3}{2}t, t, -1, 0)$ gdje su $t \in \mathbb{R}$.

(#) Riješiti sistem jednačina za razne vrijednosti parametra $\lambda \in \mathbb{R}$:

$$\begin{aligned} \lambda x_1 - 4x_2 + 9x_3 + 10x_4 &= 11 \\ 2x_1 - x_2 + 3x_3 + 4x_4 &= 5 \\ 4x_1 - 2x_2 + 5x_3 + 6x_4 &= 7 \\ 6x_1 - 3x_2 + 7x_3 + 8x_4 &= 9 \end{aligned}$$

Rj. Sistem ćemo riješiti Kroneker-Kapelijevom metodom:

$$\bar{A} = [A \mid b] = \left[\begin{array}{cccc|c} \lambda & -4 & 9 & 10 & 11 \\ 2 & -1 & 3 & 4 & 5 \\ 4 & -2 & 5 & 6 & 7 \\ 6 & -3 & 7 & 8 & 9 \end{array}\right] \overset{I_v \leftrightarrow IV_v}{\sim} \left[\begin{array}{cccc|c} 6 & -3 & 7 & 8 & 9 \\ 2 & -1 & 3 & 4 & 5 \\ 4 & -2 & 5 & 6 & 7 \\ \lambda & -4 & 9 & 10 & 11 \end{array}\right] \overset{II_v \leftrightarrow I_v}{\sim}$$

$$\sim \left[\begin{array}{cccc|c} 2 & -1 & 3 & 4 & 5 \\ 6 & -3 & 7 & 8 & 9 \\ 4 & -2 & 5 & 6 & 7 \\ \lambda & -4 & 9 & 10 & 11 \end{array}\right] \overset{I_k \leftrightarrow IV_k}{\sim} \begin{array}{cccc|c} x_4 & x_2 & x_3 & x_1 & \\ \hline 4 & -1 & 3 & 2 & 5 \\ 8 & -3 & 7 & 6 & 9 \\ 6 & -2 & 5 & 4 & 7 \\ 10 & -4 & 9 & \lambda & 11 \end{array} \overset{I_k \leftrightarrow II_k}{\sim} \begin{array}{cccc|c} x_2 & x_4 & x_3 & x_1 & \\ \hline -1 & 4 & 3 & 2 & 5 \\ -3 & 8 & 7 & 6 & 9 \\ -2 & 6 & 5 & 4 & 7 \\ -4 & 10 & 9 & \lambda & 11 \end{array}$$

$$\overset{\substack{II_v - I_v \cdot 3 \\ III_v - I_v \cdot 2 \\ IV_v - I_v \cdot 4}}{\sim} \left[\begin{array}{cccc|c} -1 & 4 & 3 & 2 & 5 \\ 0 & -4 & -2 & 0 & -6 \\ 0 & -2 & -1 & 0 & -3 \\ 0 & -6 & -3 & \lambda-8 & -9 \end{array}\right] \overset{II_k \leftrightarrow IV_k}{\sim} \begin{array}{cccc|c} x_2 & x_1 & x_3 & x_4 & \\ \hline -1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -2 & -4 & -6 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & \lambda-8 & -3 & -6 & -9 \end{array} \overset{III_v \leftrightarrow II_v}{\sim} \left[\begin{array}{cccc|c} -1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & -2 & -4 & -6 \\ 0 & \lambda-8 & -3 & -6 & -9 \end{array}\right]$$

$$\overset{\substack{III_v - II_v \cdot 2 \\ IV_v - II_v \cdot 3}}{\sim} \begin{array}{cccc|c} x_2 & x_1 & x_3 & x_4 & \\ \hline -1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda-8 & 0 & 0 & 0 \end{array}$$

a) Za $\lambda = 8$ imamo rang A = rang $\bar{A} = 2 < 4$ pa prema Kroneker-Kapelijevoj teoremi sistem ima ∞ mnogo rješenja.

2. promjenjive uzimamo proizvoljno npr. $x_4 = t$ $\quad x_1 = s$

$$\begin{aligned} -x_3 - 2x_4 &= -3 \\ -x_2 + 2x_1 + 3x_3 + 4x_4 &= 5 \end{aligned}$$

$x_3 = 3 - 2t$

$-x_2 + 2s + 3(3-2t) + 4t = 5$

$x_2 = 2s + 9 - 6t + 4t - 5$

$x_2 = 2s - 2t + 4$

Za $\lambda = 8$ rješenje sistema je $(s, 2s-2t+4, 3-2t, t)$ $t, s \in \mathbb{R}$

b) Za $\lambda \neq 8$ imamo rang A = rang $\bar{A} = 3 < 4$ pa prema Kroneker-Kapelijevom teoremu sistem ima ∞ mnogo rješenja.

1. (jednu) promjenjivu uzimamo proizvoljno npr. : $x_4 = t$

$$\begin{aligned} (\lambda - 8)x_1 &= 0 \\ -x_3 - 2x_4 &= -3 \\ -x_2 + 2x_1 + 3x_3 + 4x_4 &= 5 \end{aligned}$$

$x_1 = 0$

$x_3 = 3 - 2t$

$-x_2 + 3(3-2t) + 4t = 5$

$x_2 = 9 - 6t + 4t - 5 = -2t + 4$

Za $\lambda \neq 8$ rješenje sistema je $(0, 4-2t, 3-2t, t)$.

Homogeni sistemi linearnih jednačina

Homogeni sistem linearnih jednačina je oblika $A\cdot x = 0$ gdje je

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \quad i \quad 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$$

Teorema: Homogeni sistem ima netrivijalna rješenja akko je $D = 0$ $(\det A = D)$.

① Riješiti homogeni sistem jednačina

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \quad (1)\\ 3x_1 + x_2 - x_3 &= 0 \quad (2)\\ 2x_1 + x_2 &= 0 . \end{aligned}$$

Rj. $(1)+(2)$

$$\begin{aligned} 4x_1 + 2x_2 &= 0 \\ 2x_1 + x_2 &= 0 \quad /\cdot 2 \\ \hline 4x_1 + 2x_2 &= 0 \\ 4x_1 + 2x_2 &= 0 \\ \hline \end{aligned}$$

$4x_1 + 2x_2 = 0 \quad /:2$

$2x_1 + x_2 = 0$

sistem ima ∞ mnogo rješenji

$x_2 = -2x_1$

$x_1 = t$, $x_2 = -2t$, $t \in \mathbb{R}$,

$t - 2t + x_3 = 0$

$x_3 = t$

Sistem ima beskonačno mnogo rješenja oblika $(t, -2t, t)$

② Naći λ tako da sistem

$$\begin{aligned} 3x + y + \lambda z &= 0 \\ 4x - 8y + \lambda z &= 0 \\ 5x - 3y + 3z &= 0 \end{aligned}$$

ima netrivijalna rješenja pa nađi rješenja.

Rj.

$$D = \begin{vmatrix} 3 & 1 & \lambda \\ 4 & -8 & \lambda \\ 5 & -3 & 3 \end{vmatrix} \overset{II_v + I_v\cdot 8}{\underset{III_v + I_v\cdot 3}{=\!=}} \begin{vmatrix} 3 & 1 & \lambda \\ 28 & 0 & 9\lambda \\ 14 & 0 & 3\lambda+3 \end{vmatrix} = -\begin{vmatrix} 28 & 9\lambda \\ 14 & 3\lambda+3 \end{vmatrix} = (-14)\cdot 3 \begin{vmatrix} 2 & 3\lambda \\ 1 & \lambda+1 \end{vmatrix} = -42(-\lambda+2)$$

Za $\lambda = 2$ $(D=0)$ u sistemu postoje netrivijalna rješenja.

Sistem sad izgleda:

$$\begin{aligned} 3x + y + 2z &= 0 \quad /\cdot 3 \\ 4x - 8y + 2z &= 0 \quad /\cdot 3 \\ 5x - 3y + 3z &= 0 \quad /\cdot 2 \\ \hline \end{aligned} \qquad \begin{aligned} 9x + 3y + 6z &= 0 \quad (1) \\ 12x - 24y + 6z &= 0 \quad (2) \\ 10x - 6y + 6z &= 0 \quad (3) \\ \hline \end{aligned}$$

$(3)-(1)$: $x - 9y = 0$.

$(2)-(1)$ $\quad 3x - 27y = 0 \quad /:3$

$\overline{x - 9y = 0}$

$x = 9y$, $z = -14y$ postoji ∞ mnogo rješenja

$(9t, t, -14t)$, $t \in \mathbb{R}$ su rješenja sistema

③ Za koje vrijednosti λ sistem ima netrivijalna rješenja

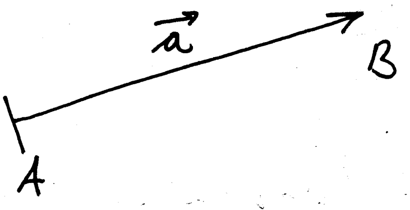
$$\begin{aligned} \lambda x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + \lambda x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 + \lambda x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 + \lambda x_4 &= 0 . \end{aligned}$$

Rj. za $\lambda = 1$ ili $\lambda = -3$

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

Vektori

Vektor definišemo kao orjentisanu duž.



$\vec{AB} = \vec{a}$

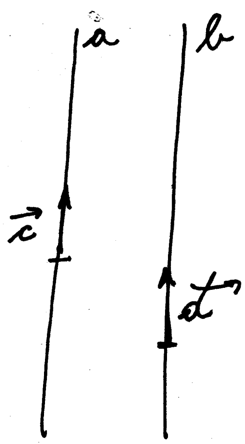
$\vec{O}$ nula vektor

Svaki vektor ima intenzitet, pravac i smijer

$|\vec{a}|$ intenzitet (veličina duži)

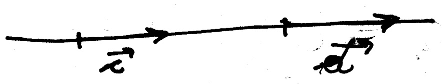
$|\vec{AB}| \geq 0 \quad \forall$ tačke A i B

Pravac vektora određena je pravom na kojoj vektor leži i tu pravu zovemo nosač vektora.

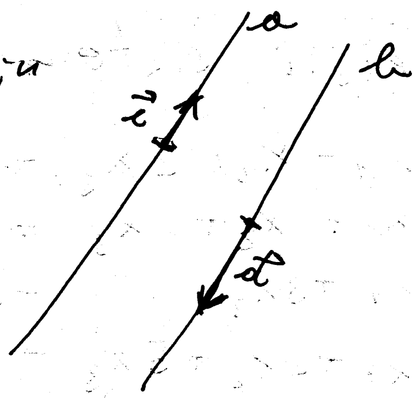


a||b -prave

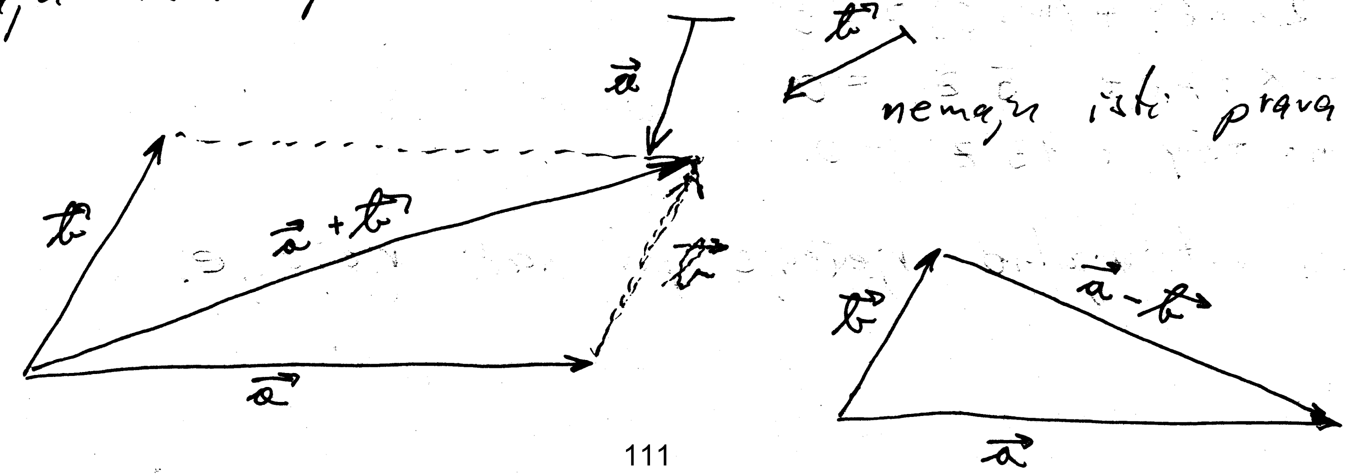
$\vec{c}$ i $\vec{d}$ vektori imaju isti pravac



a||b

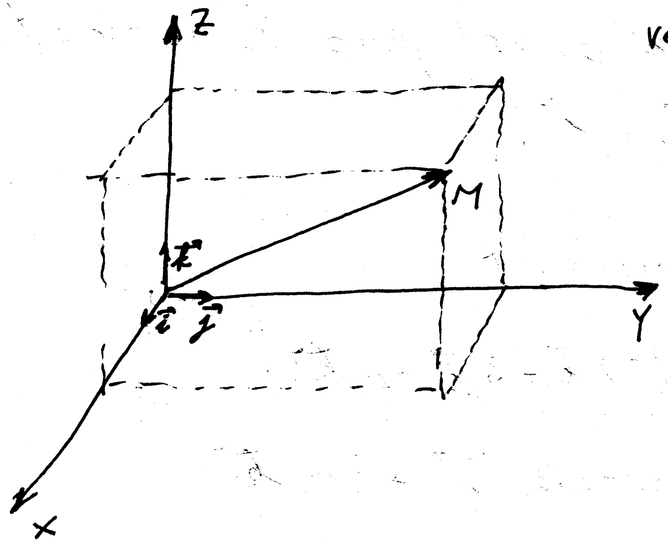


Smjer vektora određen je izborom početne i završne tačke. Vektori se mogu porediti po smjeru ako imaju isti pravac.



nemaju isti pravac

Ako je $\vec{a}$ jedinični vektor tada je $|\vec{a}|=1$.



vektor $\vec{OM}$ u koordinatnom sistemu

$$\vec{OM} = x\vec{i} + Y\vec{j} + z\vec{k}$$

$$\vec{OM}=(x, Y, z)$$

$M_1(x_1, Y_1, z_1)$

$M_2(x_2, Y_2, z_2)$

$\overrightarrow{M_1M_2}(x_2-x_1, Y_2-Y_1, z_2-z_1)$

$$\vec{a}=(a_1, a_2, a_3)$$

$$|\vec{a}|=\sqrt{a_1^2+a_2^2+a_3^2}$$

(komplanarni - nalaze se u istoj ravni)

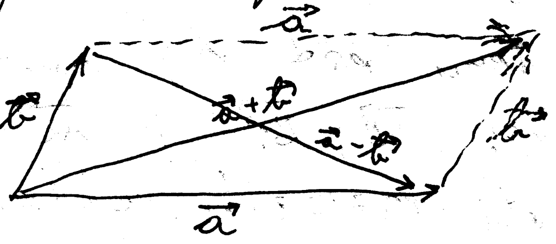
Vektori $\vec{a}$, $\vec{b}$, $\vec{c}$ su linearno zavisni ako postoje skalari α, β i γ različiti od 0 tako da važi

$$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \vec{0}$$

$\vec{a} = \lambda\vec{b} + \mu\vec{c}$ razlaganje vektora $\vec{a}$ preko ~~vektora~~ $\vec{b}$ i $\vec{c}$ (vektori se nalaze u istoj ravni)

① Kakav međusobni položaj zauzimaju vektori $\vec{a}$ i $\vec{b}$ ako je $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$.

Rj. Pretpostavimo da su vektori $\vec{a}$ i $\vec{b}$ dovedeni na zajednički početak:



Imamo paralelogram kod koga su dijagonale jednake.

Kad je ovo moguće?

Ovo je moguće samo u slučaju kad je dati paralelogram pravougaonik ili kvadrat. I u jednom i u drugom slučaju imamo da je $\vec{a} \perp \vec{b}$ ($\vec{a}$ i $\vec{b}$ su okomiti vektori).

② Ispitati linearnu zavisnost vektora $\vec{a}=(2,3,-4)$, $\vec{b}=(3,-2,0)$ i $\vec{c}=(0,1,1)$.

Rj. $\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=\vec{0}$

$\alpha(2,3,-4)+\beta(3,-2,0)+\gamma(0,1,1)=(0,0,0)$

$$\begin{aligned} 2\alpha+3\beta \quad &=0 \\ 3\alpha-2\beta+\gamma&=0 \\ -4\alpha \quad +\gamma&=0 \end{aligned}$$

$$\det M=\begin{vmatrix} 2 & 3 & 0 \\ 3 & -2 & 1 \\ -4 & 0 & 1 \end{vmatrix} \overset{III_v-II_v}{=\!=\!=} \begin{vmatrix} 2 & 3 & 0 \\ 3 & -2 & 1 \\ -7 & 2 & 0 \end{vmatrix} =$$

$$=(-1)\begin{vmatrix} 2 & 3 \\ -7 & 2 \end{vmatrix}=(-1)(4+21)=-25$$

$\det M \neq 0$

sistem ima samo trivijalna rješenja ((0,0,0))

Vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ su linearno nezavisni.

③ Dokazati da su vektori $\vec{a}=(3,1,8)$, $\vec{b}=(3,4,5)$ i $\vec{c}=(2,3,3)$ linearno zavisni.

$\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=\vec{0}$

$\alpha(3,1,8)+\beta(3,4,5)+\gamma(2,3,3)=(0,0,0)$

$$\begin{aligned} 3\alpha+3\beta+2\gamma&=0 \\ \alpha+4\beta+3\gamma&=0 \\ 8\alpha+5\beta+3\gamma&=0 \end{aligned}$$

$$\det M=\begin{vmatrix} 3 & 3 & 2 \\ 1 & 4 & 3 \\ 8 & 5 & 3 \end{vmatrix} \underset{III_v-II_v\cdot 8}{\overset{I_v-II_v\cdot 3}{=\!=\!=}} \begin{vmatrix} 0 & -9 & -7 \\ 1 & 4 & 3 \\ 0 & -27 & -21 \end{vmatrix}=(-1)\begin{vmatrix} -9 & -7 \\ -27 & -21 \end{vmatrix}$$

$$=(-1)(-9)(-7)\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix}=0$$

$\det M=0$

$\operatorname{rang} M<3$

sistem ima netrivijalna rješenja

Vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ su linearno zavisni.

④ Diskutovati linearnu zavisnost vektora $\vec{a}=(3,-8,2)$, $\vec{b}=(7,6,5)$ i $\vec{c}=(5,2,6-\lambda)$ u zavisnosti od parametra λ.

Rj. $\det M=182-74\lambda$

1° $\lambda=\frac{182}{74}$ vektori linearno zavisni

2° $\lambda\neq\frac{182}{74}$ vektori linearno nezavisni

⑤ Odrediti parametar λ tako da vektori $\vec{a} = \lambda\vec{i} + \vec{j} + 4\vec{k}$, $\vec{b} = \vec{i} - 2\lambda\vec{j}$ i $\vec{c} = 3\lambda\vec{i} - 3\vec{j} + 4\vec{k}$ budu komplanarni pa za tako dobijeno λ razložiti vektor $\vec{a}$ preko vektora $\vec{b}$ i $\vec{c}$.

Rj. $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0$ uslov komplanarnosti

$$\alpha(\lambda, 1, 4) + \beta(1, -2\lambda, 0) + \gamma(3\lambda, -3, 4) = (0, 0, 0)$$

$$\begin{aligned} \lambda\alpha + \beta + 3\lambda\gamma &= 0 \\ \alpha - 2\lambda\beta - 3\gamma &= 0 \\ 4\alpha + \quad 4\gamma &= 0 \end{aligned}$$

sistem, α, β i γ su nepoznate

$$D = \begin{vmatrix} \lambda & 1 & 3\lambda \\ 1 & -2\lambda & -3 \\ 4 & 0 & 4 \end{vmatrix} \xlongequal{III_k - I_k} \begin{vmatrix} \lambda & 1 & 2\lambda \\ 1 & -2\lambda & -4 \\ 4 & 0 & 0 \end{vmatrix} =$$

$$= 4\begin{vmatrix} 1 & 2\lambda \\ -2\lambda & -4 \end{vmatrix} = 4 \cdot 2\begin{vmatrix} 1 & 2\lambda \\ -\lambda & -2 \end{vmatrix} = 8 \cdot 2\begin{vmatrix} 1 & \lambda \\ -\lambda & -1 \end{vmatrix}$$

$$D = 16(-1 + \lambda^2) = 16(\lambda^2 - 1)$$

Za $\lambda = \pm 1$ imamo da je $D = 0$ $\Rightarrow$ sistem ima beskonačno mnogo rješenja (za $\lambda = \pm 1$).

Za $\lambda = \pm 1$ vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ su komplanarni: Uzmimo da je $\lambda = 1$:

$\vec{a} = (1, 1, 4)$
$\vec{b} = (1, -2, 0)$
$\vec{c} = (3, -3, 4)$

$$\vec{a} = \alpha\vec{b} + \beta\vec{c}$$

$$\alpha(1, -2, 0) + \beta(3, -3, 4) = (1, 1, 4)$$

$$\begin{aligned} \alpha + 3\beta &= 1 \\ -2\alpha - 3\beta &= 1 \\ 4\beta &= 4 \end{aligned}$$

$\beta = 1$
$\alpha + 3 = 1$
$\alpha = -2$

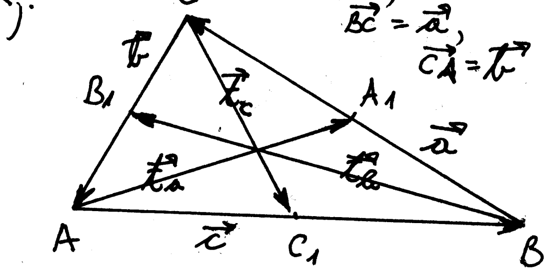
za $\lambda = 1$
$\vec{a} = -2\vec{b} + \vec{c}$
razlaganje vektora $\vec{a}$ preko vektora $\vec{b}$ i $\vec{c}$

Za $\lambda = -1$ vektor $\vec{a}$ razložen preko vektora $\vec{b}$ i $\vec{c}$:

$$\vec{a} = 2\vec{b} + \vec{c}$$

⑥ Stranice trougla su vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$. Pomoću ovih vektora izraziti težišne linije trougla (vidi sliku).

Rj.



$\overrightarrow{AB} = \vec{c}$,
$\overrightarrow{BC} = \vec{a}$,
$\overrightarrow{CA} = \vec{b}$

Težišna linija je duž koja spaja tjeme trougla sa sredinom stranice nasprem tog tjemena.

$$\vec{t_a} = \overrightarrow{AA_1} = \overrightarrow{AB} + \overrightarrow{BA_1} = \vec{c} + \frac{1}{2}\vec{a}$$

$$\vec{t_a} = \overrightarrow{AA_1} = \overrightarrow{AC} + \overrightarrow{CA_1} = -\overrightarrow{CA} + \frac{1}{2}\overrightarrow{BC} = -\vec{b} - \frac{1}{2}\vec{a}$$

Za vježbu: $\vec{t_b} = \vec{a} + \frac{1}{2}\vec{b} = -\vec{c} - \frac{1}{2}\vec{b}$, $\quad \vec{t_c} = \vec{b} - -\vec{c} = -\vec{a} - \frac{1}{2}\vec{c}$

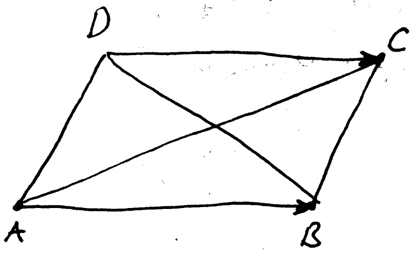
⑦ Data su tjemena paralelograma $\square ABCD$
$A(-3, 2, \lambda)$, $B(3, -3, 1)$ i $C(5, \lambda, 2)$.

a) Odrediti tjeme D

b) Odrediti λ tako da je $|\overrightarrow{AD}|=\sqrt{14}$

c) Za veću vrijednost λ (nađenu pod b) ispitati linearnu zavisnost vektora: $\overrightarrow{AD}$, $\overrightarrow{BD}$ i $\overrightarrow{AC}$. U slučaju linearne zavisnosti razložiti vektor $\overrightarrow{AC}$ preko $\overrightarrow{AD}$ i $\overrightarrow{BD}$

Rj.



a) $D=?$

Šta znamo za paralelogram?
Paralelogram ima dva para naspremnih podudarnih stranica, pa:
$\overrightarrow{AD}=\overrightarrow{BC}$ i $\overrightarrow{AB}=\overrightarrow{DC}$

$$\left.\begin{array}{l} A(-3,2,\lambda) \\ D(x,y,z) \end{array}\right\} \Rightarrow \overrightarrow{AD}(x+3, y-2, z-\lambda)$$

$$\left.\begin{array}{l} B(3,-3,1) \\ C(5,\lambda,2) \end{array}\right\} \Rightarrow \overrightarrow{BC}(2, \lambda+3, 1)$$

$$\Rightarrow \begin{array}{c} x+3=2 \\ y-2=\lambda+3 \\ z-\lambda=1 \\ \hline x=-1 \\ y=\lambda+5 \\ z=\lambda+1 \end{array}$$

$D(-1, \lambda+5, \lambda+1)$

II način: posmatramo sredine dijagonala
(ostavljamo studentima za vježbu)

b) $\lambda=?$ $\overrightarrow{AD}=\sqrt{14}$

$$\left.\begin{array}{l} A(-3,2,\lambda) \\ D(-1, \lambda+5, \lambda+1) \end{array}\right\} \Rightarrow \overrightarrow{AD}(2, \lambda+3, 1)$$

$$\begin{array}{c} |\overrightarrow{AD}|=\sqrt{4+(\lambda+3)^2+1} \\ |\overrightarrow{AD}|=\sqrt{14} \\ \hline 4+\lambda^2+6\lambda+9+1=14 \end{array}$$

$$\begin{array}{c} \lambda^2+6\lambda=0 \\ \lambda(\lambda+6)=0 \\ \hline \end{array} \qquad \begin{array}{l} \lambda_1=0 \\ \lambda_2=-6 \end{array}$$

Za $\lambda=0$ ili $\lambda=-6$ imamo $\overrightarrow{AD}=\sqrt{14}$.

c)ˇ $\lambda=0$ Rj: $\overrightarrow{AC}=2\overrightarrow{AD}-\overrightarrow{BD}$

razlaganje vektora $\overrightarrow{AC}$

Skalarni proizvod

(dva vektora)

$$\vec a\cdot\vec b=|\vec a|\cdot|\vec b|\cdot\cos\angle(\vec a,\vec b) \Rightarrow \cos\angle(\vec a,\vec b)=\frac{\vec a\cdot\vec b}{|\vec a|\cdot|\vec b|}$$

$\vec a(x_1, y_1, z_1)$

$\vec b(x_2, y_2, z_2)$

$$\vec a\cdot\vec b = x_1\cdot x_2+y_1\cdot y_2+z_1\cdot z_2$$

za $\vec a\cdot\vec b=0$ vekori $\vec a$ i $\vec b$ su okomiti

①. Dati su vektori $\vec a=(1, 2, 1)$ i $\vec b=(2, 1, -1)$. Izračunati $\vec a\cdot\vec b$, $(\vec a-\vec b)^2$, $\sqrt{\vec a^2}$ i $\angle(\vec a,\vec b)$.

Rj: $\vec a\cdot\vec b=|\vec a||\vec b|\cos\angle(\vec a,\vec b)$

$\vec a\cdot\vec b=(1,2,1)(2,1,-1)=2+2-1=3$ $\qquad$ $\vec a\cdot\vec b=3$

$\vec a=(1,2,1)$ $\qquad$ $\vec a-\vec b=(-1,1,2)$

$\vec b=(2,1,-1)$ $\qquad$ $(\vec a-\vec b)^2=(-1,1,2)(-1,1,2)=1+1+4=6$ $\qquad$ $(\vec a-\vec b)^2=6$

$\vec a^2=(1,2,1)(1,2,1)=1+4+1=6$

$\sqrt{\vec a^2}=\sqrt{6}$ $\qquad$ $|\vec a|=\sqrt{\vec a^2}=\sqrt{6}$, $|\vec b|=\sqrt{4+1+1}=\sqrt{6}$

$\cos\angle(\vec a,\vec b)=\frac{\vec a\cdot\vec b}{|\vec a||\vec b|}=\frac{3}{\sqrt6\cdot\sqrt6}=\frac{3}{6}=\frac{1}{2}$ $\Rightarrow$ $\angle(\vec a,\vec b)=60°$

ugao između vektora $\vec a$ i $\vec b$.

②. Odrediti parametar λ tako da vektori $\vec a(2a^\lambda, \lambda, \lambda-1)$ i $\vec b(\lambda+1, \lambda-2, 0)$ imaju isti intenzitet a zatim naći ugao između njih.

Rj. $|\vec a|=|\vec b|$

$$\left.\begin{array}{l}|\vec a|=\sqrt{(2a^\lambda)^2+\lambda^2+(\lambda-1)^2}\\ |\vec b|=\sqrt{(\lambda+1)^2+(\lambda-2)^2+0^2}\end{array}\right\}\Nearrow$$

$4a^{2\lambda}+\cancel{\lambda^2}+\cancel{\lambda^2}\underline{-2\lambda}+\underline{1}=$

$=\cancel{\lambda^2}\underline{+2\lambda}\underline{\underline{+1}}+\cancel{\lambda^2}\underline{-4\lambda}+4$

$4a^{2\lambda}=4$

$a^{2\lambda}=1$

$$a^{2\lambda} = a^{0} \Rightarrow 2\lambda = 0$$

$$\lambda = 0$$

Za $\lambda = 0$ vektori $\vec{a}$ i $\vec{b}$ imaju isti intenzitet.

$$\left.\begin{matrix} \vec{a}(2,0,-1) \\ \vec{b}(1,-2,0) \end{matrix}\right\} \Rightarrow \vec{a}\cdot\vec{b} = 2+0+0 = 2$$

$$\cos\measuredangle(\vec{a},\vec{b}) = \frac{2}{\sqrt{5}\cdot\sqrt{5}} = \frac{2}{5}$$

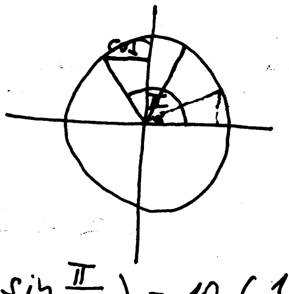
$$\measuredangle(\vec{a},\vec{b}) = \arccos\frac{2}{5}$$ ugao između vektora

3. Zadani su vektori $\vec{p} = \lambda\vec{a} + 17\vec{b}$ i $\vec{q} = 3\vec{a} - \vec{b}$ gdje je $|\vec{a}| = 2$, $|\vec{b}| = 5$ a $\measuredangle(\vec{a},\vec{b}) = \frac{2\pi}{3}$ (ugao između vektora $\vec{a}$ i $\vec{b}$). Odrediti koeficijent λ tako da vektori $\vec{p}$ i $\vec{q}$ budu međusobno okomiti.

Rj. $\vec{p}\cdot\vec{q} = 0$ (uslov okomitosti)

$$\vec{p}\cdot\vec{q} = (\lambda\vec{a} + 17\vec{b})(3\vec{a} - \vec{b}) = 3\lambda\vec{a}^{2} - \lambda\vec{a}\vec{b} + 51\vec{a}\vec{b} - 17\vec{b}^{2}$$

$$= 3\lambda\vec{a}^{2} + (51-\lambda)\vec{a}\vec{b} - 17\vec{b}^{2}$$



$$\vec{a}^{2} = \vec{a}\cdot\vec{a} = |\vec{a}|\cdot|\vec{a}|\cdot\cos\measuredangle(\vec{a},\vec{a}) = 2\cdot 2\cdot\cos 0^\circ = 4$$

$$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\measuredangle(\vec{a},\vec{b}) = 2\cdot 5\cdot\cos\frac{2\pi}{3} = 10\cdot\left(-\sin\frac{\pi}{6}\right) = 10\cdot\left(-\frac{1}{2}\right) = -5$$

$$\vec{b}^{2} = \vec{b}\cdot\vec{b} = |\vec{b}|\cdot|\vec{b}|\cdot\cos\measuredangle(\vec{b},\vec{b}) = 5\cdot 5\cdot\cos 0^\circ = 25$$

$$\vec{p}\cdot\vec{q} = 0$$

$$3\lambda\vec{a}^{2} + (51-\lambda)\vec{a}\cdot\vec{b} - 17\vec{b}^{2} = 0$$

$$3\lambda\cdot 4 + (51-\lambda)\cdot(-5) - 17\cdot 25 = 0$$

$$12\lambda - 225 + 5\lambda - 425 = 0$$

$$17\lambda - 680 = 0$$

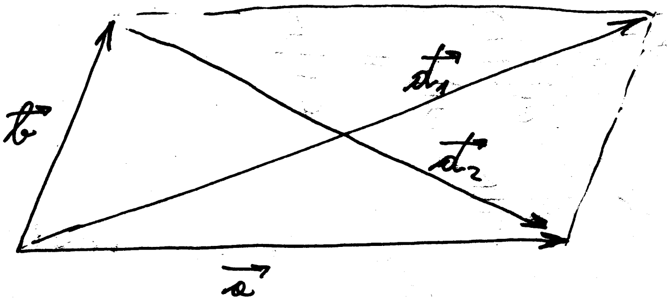
$$17\lambda = 680$$

$$\lambda = 40$$

Za $\lambda = 40$ vektori $\vec{p}$ i $\vec{q}$ su međusobno okomiti.

④ Naći dužine dijagonala i ugao između njih, paralelograma konstruisanog nad vektorima $\vec{a} - 2\vec{m} + \vec{n}$ i $\vec{b} = \vec{m} - 2\vec{n}$, gdje su $\vec{m}$ i $\vec{n}$ jedinični vektori koji obrazuju ugao od $\frac{\pi}{3}$.

Rj.



$\vec{a} = 2\vec{m} + \vec{n}$

$\vec{b} = \vec{m} - 2\vec{n}$

$\vec{d_1} = \vec{a} + \vec{b}$

$\vec{d_2} = \vec{a} - \vec{b}$

$|\vec{d_1}| = ?$ $|\vec{d_2}| = ?$ $\measuredangle(\vec{d_1}, \vec{d_2}) = ?$

$\vec{m}$ i $\vec{n}$ su jedinični vektori $\Rightarrow |\vec{m}| = |\vec{n}| = 1$

$\vec{m} \cdot \vec{n} = |\vec{m}| \cdot |\vec{n}| \cdot \cos\measuredangle(\vec{m}, \vec{n}) = 1 \cdot 1 \cdot \cos\frac{\pi}{3} = \frac{1}{2}$

$\vec{a} + \vec{b} = 3\vec{m} - \vec{n}$

$|\vec{a} + \vec{b}| = \sqrt{(3\vec{m} - \vec{n})^2} = \sqrt{9\vec{m}^2 - 6\vec{m}\vec{n} + \vec{n}^2} = \sqrt{9 - 3 + 1} = \sqrt{7}$

$\vec{a} - \vec{b} = \vec{m} + 3\vec{n}$

$|\vec{a} - \vec{b}| = \sqrt{(\vec{m} + 3\vec{n})^2} = \sqrt{\vec{m}^2 + 6\vec{m} \cdot \vec{n} + 9\vec{n}^2} = \sqrt{1 + 3 + 1} = \sqrt{13}$

$\vec{d_1} \cdot \vec{d_2} = |\vec{d_1}| \cdot |\vec{d_2}| \cdot \cos\measuredangle(\vec{d_1}, \vec{d_2})$

$$\cos\measuredangle(\vec{d_1}, \vec{d_2}) = \frac{\vec{d_1} \cdot \vec{d_2}}{|\vec{d_1}| \cdot |\vec{d_2}|}$$

$\vec{d_1} \cdot \vec{d_2} = (\vec{a} + \vec{b})(\vec{a} - \vec{b}) = \vec{a}^2 - \vec{b}^2 = |\vec{a}|^2 - |\vec{b}|^2$

$|\vec{a}| = \sqrt{(2\vec{m} + \vec{n})^2} = \sqrt{4\vec{m}^2 + 4\vec{m}\vec{n} + \vec{n}^2} = \sqrt{4 + 2 + 1} = \sqrt{7}$

$|\vec{b}| = \sqrt{(\vec{m} - 2\vec{n})^2} = \sqrt{\vec{m}^2 - 4\vec{m}\vec{n} + 4\vec{n}^2} = \sqrt{1 - 2 + 4} = \sqrt{3}$

$\vec{d_1} \cdot \vec{d_2} = 7 - 3 = 4$ $\cos\measuredangle(\vec{d_1}, \vec{d_2}) = \frac{4}{\sqrt{91}}$

Dijagonale $\vec{d_1}$ i $\vec{d_2}$ paralelograma imaju dužine $\sqrt{7}$ i $\sqrt{13}$ a obrazuju ugao od $\arccos\frac{4}{\sqrt{91}}$.

(#) Dati su vektori $\vec{a}=(8-\lambda, 3, -1-\lambda)$, $\vec{b}=(7,1,0)$ i $\vec{c}=(7,7,0)$. Odrediti parametar λ tako da $\sphericalangle(\vec{a},\vec{b})=\sphericalangle(\vec{a},\vec{c})$ (da ugao između vektora $\vec{a}$ i $\vec{b}$ bude jednak uglu između vektora $\vec{a}$ i $\vec{c}$), pa za dobijenu vrijednost λ odrediti veličinu ugla.

Rj.

$$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos\sphericalangle(\vec{a},\vec{b})$$

$$\vec{a}\cdot\vec{b}=(8-\lambda, 3, -1-\lambda)\cdot(7,1,0)=56-7\lambda+3=59-7\lambda$$

$$|\vec{a}|=\sqrt{(8-\lambda)^2+3^2+(-1-\lambda)^2}$$

$$|\vec{b}|=\sqrt{49+1}=\sqrt{50}=5\sqrt{2}$$

$$\cos\sphericalangle(\vec{a},\vec{b})=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$$

$$|\vec{c}|=\sqrt{49+49}=7\sqrt{2}$$

$$\cos\sphericalangle(\vec{a},\vec{c})=\frac{\vec{a}\cdot\vec{c}}{|\vec{a}|\cdot|\vec{c}|}$$

Kako tražimo λ tako da $\sphericalangle(\vec{a},\vec{b})=\sphericalangle(\vec{a},\vec{c}) \Rightarrow$

$$\Rightarrow \cos\sphericalangle(\vec{a},\vec{b})=\cos\sphericalangle(\vec{a},\vec{c}) \Rightarrow \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}=\frac{\vec{a}\cdot\vec{c}}{|\vec{c}|}$$

$$\vec{a}\cdot\vec{c}=(8-\lambda, 3, -1-\lambda)(7,7,0)=56-7\lambda+21=77-7\lambda$$

$$\frac{59-7\lambda}{5\sqrt{2}}=\frac{77-7\lambda}{7\sqrt{2}} \quad /\cdot 35\sqrt{2}$$

$$413-49\lambda=385-35\lambda$$

$$14\lambda=28$$

$$\lambda=2$$

Za vrijednost $\lambda=2$ imamo $\sphericalangle(\vec{a},\vec{b})=\sphericalangle(\vec{a},\vec{c})$

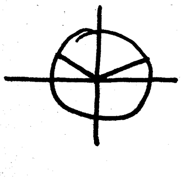
$$\lambda=2 \Rightarrow \vec{a}=(6,3,-3)$$

$$|\vec{a}|=\sqrt{36+9+9}=\sqrt{54}=3\sqrt{6}$$

$$\cos\sphericalangle(\vec{a},\vec{b})=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}=\frac{(6,3,-3)\cdot(7,1,0)}{3\sqrt{6}\cdot5\sqrt{2}}=\frac{42+3}{15\sqrt{12}}=\frac{45}{15\sqrt{4\cdot3}}=$$

$$=\frac{3}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{3}}{2}$$

$$\cos\sphericalangle(\vec{a},\vec{b})=\frac{\sqrt{3}}{2} \Rightarrow$$

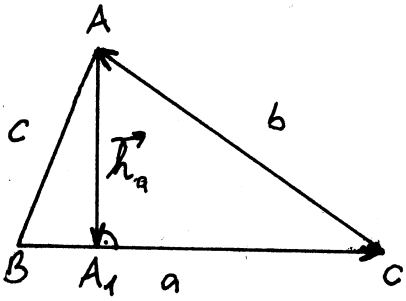


$$\sphericalangle(\vec{a},\vec{b})=\frac{\pi}{6}=30^\circ \text{ ili } \sphericalangle(\vec{a},\vec{b})=\frac{11\pi}{6}=330^\circ$$

veličina ugla

(#) Odrediti vektor visine $\vec{h}_a$ iz vrha A trougla ΔABC ako je $\vec{BC}=\vec{m}+2\vec{n}$, $\vec{CA}=2\vec{m}-\vec{n}$, $|\vec{m}|=|\vec{n}|=\sqrt{3}$, $\sphericalangle(\vec{m},\vec{n})=\frac{\pi}{2}$.

Rj.



$$\vec{AB}=\vec{BC}+\vec{CA}=\vec{m}+2\vec{n}+2\vec{m}-\vec{n}=3\vec{m}+\vec{n}$$

$\vec{h}_a=?$

$\vec{h}_a=x\vec{m}+y\vec{n}$

$\vec{h}_a\perp\vec{BC}\Rightarrow\vec{h}_a\cdot\vec{BC}=0$ tj.

$$(x\vec{m}+y\vec{n})\cdot(\vec{m}+2\vec{n})=x\vec{m}^2+2x\vec{m}\cdot\vec{n}+y\vec{m}\vec{n}+y\vec{n}^2\overset{(*)}{=}$$

$\vec{m}^2=|\vec{m}|^2=3$

$\vec{n}^2=|\vec{n}|^2=3$ $\quad\ldots(*)$

$\vec{m}\cdot\vec{n}=|\vec{m}||\vec{n}|\cos\sphericalangle(\vec{m},\vec{n})=0$

$\overset{(*)}{=}3x+3y=0$

tj. $x+y=0$

$x=-y$

$P_{\Delta ABC}=\sqrt{s(s-a)(s-b)(s-c)}$

$P_{\Delta ABC}=\dfrac{|\vec{h}_a|\cdot|\vec{BC}|}{2}$

$a^2=|\vec{BC}|^2=\vec{BC}^2=(\vec{m}+2\vec{n})^2=$

$=\vec{m}^2+4\vec{m}\vec{n}+4\vec{n}^2=3+4\cdot3=15$

$b^2=|\vec{CA}|^2=\vec{CA}^2=(2\vec{m}-\vec{n})^2=4\vec{m}^2-4\vec{m}\vec{n}+\vec{n}^2$

$=15$

$c^2=|\vec{AB}|^2=\vec{AB}^2=(3\vec{m}+\vec{n})^2=9\vec{m}^2+6\vec{m}\vec{n}+\vec{n}^2=30$

$a=\sqrt{15},\ b=\sqrt{15},\ c=\sqrt{30}$

$$s=\frac{a+b+c}{2}=\frac{2\sqrt{15}+\sqrt{30}}{2}=\sqrt{15}+\frac{\sqrt{2}}{2}\sqrt{15}=\sqrt{15}+\frac{\sqrt{30}}{2}$$

$$P_{\Delta ABC}=\sqrt{\left(\sqrt{15}+\frac{\sqrt{2}}{2}\sqrt{15}\right)\left(\frac{\sqrt{2}}{2}\sqrt{15}\right)\left(\frac{\sqrt{2}}{2}\sqrt{15}\right)\left(\sqrt{15}-\frac{\sqrt{30}}{2}\right)}=$$

$$=\sqrt{\left(15-\frac{30}{4}\right)\cdot\frac{1}{4}\cdot15}=\sqrt{\frac{30}{4}\cdot\frac{1}{4}\cdot15}=\frac{15}{4}\sqrt{2}\quad\ldots(\Delta)$$

$$P_{\Delta ABC}=\frac{|\vec{h}_a|\cdot\sqrt{15}}{2}\overset{(\Delta)}{\Longrightarrow}|\vec{h}_a|\cdot\sqrt{15}=\frac{15}{2}\sqrt{2}\rightarrow|\vec{h}_a|=\frac{15}{2}\sqrt{\frac{2}{15}}$$

$$|\vec{h}_a|^2=\frac{15^2}{2^2}\cdot\frac{2}{15}=\frac{15}{2}=\vec{h}_a^2=(x\vec{m}+y\vec{n})^2=x^2\vec{m}^2+2xy\vec{m}\vec{n}+y^2\vec{n}^2$$

$=3x^2+3y^2$ tj. $3x^2+3y^2=\frac{15}{2}\Rightarrow x^2+y^2=\frac{5}{2}$ kako $x=-y$

$2y^2=\frac{5}{2}$ $\quad y_{1,2}=\pm\frac{\sqrt{5}}{2}$ $\quad y_1=\frac{\sqrt{5}}{2}\Rightarrow x_1=-\frac{\sqrt{5}}{2}$ $\quad\vec{h}_a=\pm\left(\frac{\sqrt{5}}{2}\vec{m}-\frac{\sqrt{5}}{2}\vec{n}\right)$

$y^2=\frac{5}{4}$ $\quad y_2=-\frac{\sqrt{5}}{2}\Rightarrow x_2=\frac{\sqrt{5}}{2}$ $\quad\pm$ zavisi od $\vec{AA_1}$ ili $\vec{A_1A}$

(#) Dati su vektori $\vec{a}=(\lambda,-\lambda-1,-\lambda-2)$, $\vec{b}=(2,-1,-7)$ i $\vec{c}=(6,-3,-3)$. Odrediti parametar λ tako da $\sphericalangle(\vec{a},\vec{b})=\sphericalangle(\vec{a},\vec{c})$ (ugao između vektora $\vec{a}$ i $\vec{b}$ bude jednak uglu između vektora $\vec{a}$ i $\vec{c}$), pa za dobijenu vrijednost λ odrediti veličinu ugla.

Rj.

$\vec{a}=(\lambda,-\lambda-1,-\lambda-2)$

$\vec{b}=(2,-1,-7)$

$\vec{c}=(6,-3,-3)$

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\sphericalangle(\vec{a},\vec{b})$$

$$\cos\sphericalangle(\vec{a},\vec{b})=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

isto tako

$$\cos\sphericalangle(\vec{a},\vec{c})=\frac{\vec{a}\cdot\vec{c}}{|\vec{a}||\vec{c}|}$$

Imamo $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{\vec{a}\cdot\vec{c}}{|\vec{a}||\vec{c}|} \Rightarrow \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}=\frac{\vec{a}\cdot\vec{c}}{|\vec{c}|}$

$$\left.\begin{aligned}
\vec{a}\cdot\vec{b}&=2\lambda+\lambda+1+7\lambda+14=10\lambda+15\\
\vec{a}\cdot\vec{c}&=6\lambda+3\lambda+3+3\lambda+6=12\lambda+9\\
|\vec{b}|&=\sqrt{4+1+49}=\sqrt{54}=\sqrt{6\cdot 9}=3\sqrt{6}\\
|\vec{c}|&=\sqrt{36+9+9}=\sqrt{54}=3\sqrt{6}
\end{aligned}\right\} \Rightarrow \frac{10\lambda+15}{3\sqrt{6}}=\frac{12\lambda+9}{3\sqrt{6}}$$

$$10\lambda-12\lambda=9-15$$

$$2\lambda=6$$

$$\lambda=3$$

tražena vrijednost za λ

$\vec{a}=(3,-4,-5)$

$|\vec{a}|=\sqrt{9+16+25}=\sqrt{50}=\sqrt{2\cdot 25}=5\sqrt{2}$

$|\vec{b}|=3\sqrt{6}$

$\vec{a}\cdot\vec{b}=30+15=45$

$$\cos\sphericalangle(\vec{a},\vec{b})=\frac{45}{5\sqrt{2}\cdot 3\sqrt{6}}=\frac{3}{\sqrt{2}\cdot\sqrt{2\cdot 3}}$$

$$\cos\sphericalangle(\vec{a},\vec{b})=\frac{3}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{3\sqrt{3}}{2\cdot 3}=\frac{\sqrt{3}}{2}$$

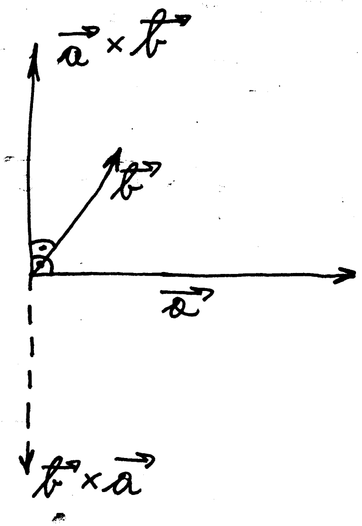
$$\cos\sphericalangle(\vec{a},\vec{b})=\frac{\sqrt{3}}{2} \Rightarrow \sphericalangle(\vec{a},\vec{b})=30^\circ$$

veličina ugla između vektora

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

Vektorski proizvod

(dva vektova)



$\vec{a}\cdot\vec{b}$ = realan broj

$\vec{a}\times\vec{b}$ = vektor

$$|\vec{a}\times\vec{b}| = |\vec{a}|\cdot|\vec{b}|\cdot\sin\sphericalangle(\vec{a},\vec{b})$$

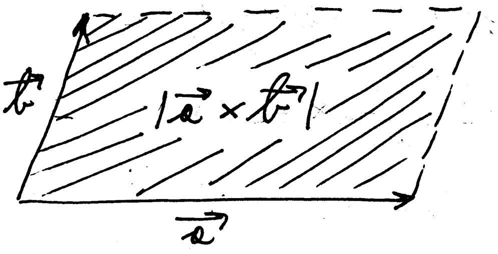
$$\vec{a}\times\vec{b}\perp\vec{a}$$

$$\vec{a}\times\vec{b}\perp\vec{b}$$

$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$$

$$\vec{a}\parallel\vec{b} \iff \vec{a}\times\vec{b} = 0$$

ovo je uslov kolinearnosti dva vektora



$$P_{\square} = |\vec{a}\times\vec{b}|$$

$\vec{a}(a_1, a_2, a_3)$

$\vec{b}(b_1, b_2, b_3)$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

① Dati su vektori $\vec{a}=(0, 2\lambda, \lambda)$, $\vec{b}=(2, 2, 1)$ i $\vec{c}=(-1, -2, -1)$. Odrediti vektor $\vec{d}$ koji zadovoljava uvjete

$$\vec{a}\times\vec{b} = \vec{c}\times\vec{d} \quad i \quad \vec{a}\times\vec{c} = \vec{b}\times\vec{d}.$$

Rj.
$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2\lambda & \lambda \\ 2 & 2 & 1 \end{vmatrix} = (2\lambda-2\lambda)\vec{i} - (0-2\lambda)\vec{j} + (0-4\lambda)\vec{k}$$
$$= (0, 2\lambda, -4\lambda)$$

Neka je vektor $\vec{d}=(x, y, z)$

$$\vec{c}\times\vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -2 & -1 \\ x & y & z \end{vmatrix} = (-2z+y)\vec{i} - (-z-x)\vec{j} + (-y-2x)\vec{k}$$

kako je $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ to je

$$\begin{array}{l} -2z+Y=0 \\ z+x=2\lambda \\ -2x-Y=-4\lambda \\ \hline Y=2z \\ z+x=2\lambda \\ -2x-Y=-4\lambda \\ \hline \end{array}$$

$$\begin{array}{l} Y=2z \\ x+z=2\lambda \\ -2x-2z=-4\lambda \quad /:(-2) \\ \hline Y=2z \\ x+z=2\lambda \quad \dots(*) \end{array}$$

$$\vec{a}\times\vec{c}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2\lambda & \lambda \\ -1 & -2 & -1 \end{vmatrix}=(-2\lambda+2\lambda)\vec{i}-(0+\lambda)\vec{j}+(0+2\lambda)\vec{k}=(0,-\lambda,2\lambda)$$

$$\vec{b}\times\vec{d}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 1 \\ x & Y & z \end{vmatrix}=(2z-Y)\vec{i}-(2z-x)\vec{j}+(2Y-2x)\vec{k}=$$

$$=(-Y+2z,\ x-2z,\ -2x+2Y)$$

kako je $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$ to je

$$\begin{array}{l} -Y+2z=0 \\ x \quad -2z=-\lambda \\ -2x+2Y \quad =2\lambda \\ \hline Y=2z \\ x-2z=-\lambda \\ -2x+4z=2\lambda \quad /:(-2) \\ \hline \end{array}$$

$$\begin{array}{l} Y=2z \\ x-2z=-\lambda \quad \dots(**) \end{array}$$

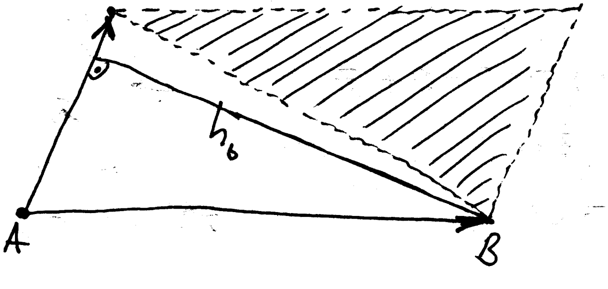
Iz (*) i (**) dobijemo

$$\begin{array}{l} Y=2z \\ -\begin{cases} x-2z=-\lambda \\ x+z=2\lambda \end{cases} \\ \hline Y=2z \\ -3z=-3\lambda \\ \hline Y=2z \\ z=\lambda,\quad Y=2\lambda,\quad x=\lambda \end{array}$$

Vektor $\vec{d}$ je $\vec{d}(\lambda,2\lambda,\lambda)$.

② Naći površinu i visinu koja odgovara stranici AC trougla $\triangle ABC$ ako je $A(-3,-2,0)$, $B(3,-3,1)$ i $C(5,0,2)$.

Rj.



$$\left.\begin{array}{l} A(-3,-2,0) \\ B(3,-3,1) \end{array}\right\} \overrightarrow{AB}(6,-1,1)$$

$$C(5,0,2) \qquad \overrightarrow{AC}(8,2,2)$$

$$P_{\square}=|\overrightarrow{AB}\times\overrightarrow{AC}|$$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & -1 & 1 \\ 8 & 2 & 2 \end{vmatrix} = (-2-2)\vec{i} - (12-8)\vec{j} + (12+8)\vec{k} = (-4,-4,20)$$

$$|\vec{AB}\times\vec{AC}| = \sqrt{16+16+400} = \sqrt{432} = \sqrt{16\cdot 27} = \sqrt{4^2\cdot 3^2\cdot 3} = 12\sqrt{3}$$

$$P_{\Delta ABC} = \frac{|\vec{AB}\times\vec{AC}|}{2} = 6\sqrt{3}$$

$$P_{\Delta ABC} = \frac{|\vec{AC}|\cdot h_b}{2} \qquad \vec{AC}(8,2,2)$$

$$|\vec{AC}| = \sqrt{64+4+4} = \sqrt{72} = \sqrt{8\cdot 9} = 6\sqrt{2}$$

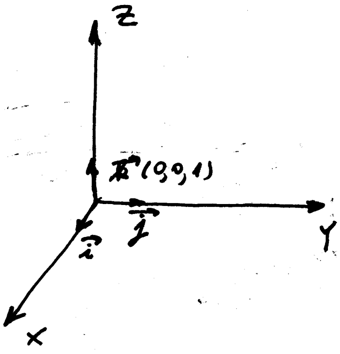
$$6\sqrt{2}\cdot h_b = 12\sqrt{3} \quad /:6\sqrt{2}$$

$$h_b = 2\sqrt{\frac{3}{2}}$$

Površina trougla ΔABC je $6\sqrt{3}$ a visina koja odgovara stranici AC iznosi $2\sqrt{\frac{3}{2}}$.

③ Vektor $\vec{r}$ je normalan na Oz osu i na vektor $\vec{a}(8,-15,3)$. Ako je $|\vec{r}|=51$ i $\sphericalangle(\vec{r},Ox)$ oštar, naći vektor $\vec{r}$.

Rj.



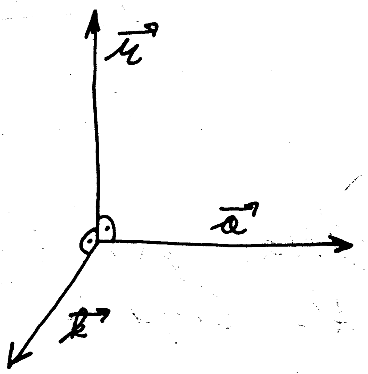
$\vec{r} \perp Oz$-osu

$\vec{r} \perp \vec{a}$

$|\vec{r}| = 51$

$\sphericalangle(\vec{r},Ox)$ oštar

$\vec{r} = ?$



$\Downarrow$

$\vec{r} \parallel \vec{a}\times\vec{k}$

tj.

$\vec{r} = \lambda(\vec{a}\times\vec{k})$

Stavimo $\vec{r}(x,y,z)$

$\vec{r} \perp Oz$-osa $\Rightarrow$ $\vec{r}\cdot\vec{k} = 0$

$$(x,y,z)(0,0,1) = 0+0+z$$

$$z=0$$

$\vec{r} \perp \vec{a}$ $\Rightarrow$ $(x,y,0)(8,-15,3) = 0$

$$8x - 15y = 0$$

$$\vec{a}\times\vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 8 & -15 & 3 \\ 0 & 0 & 1 \end{vmatrix} = -15\vec{i} - 8\vec{j}$$

$$\left.\begin{array}{l} \vec{r} = \lambda(-15,-8,0) = (-15\lambda,-8\lambda,0) \\ |\vec{r}| = 51 \end{array}\right\} \Longrightarrow \sqrt{225\lambda^2 + 64\lambda^2} = 51$$

$$\sqrt{289\lambda^2} = 51$$

$$17|\lambda| = 51$$

$$|\lambda| = 3$$

$$\lambda_1 = -3 \quad \lambda_2 = 3$$

$\vec{r}_1\ (45, 24, 0)$

$\vec{r}_2\ (-45, -24, 0)$

$\measuredangle(\vec{r}, O_x)$ oštar $\Rightarrow$ $\cos\measuredangle(\vec{r}, O_x) > 0$

tj. $\vec{r}\cdot\vec{i} > 0$

$\vec{r}\cdot\vec{i} = (x, y, z)(0, 1, 0) = y$

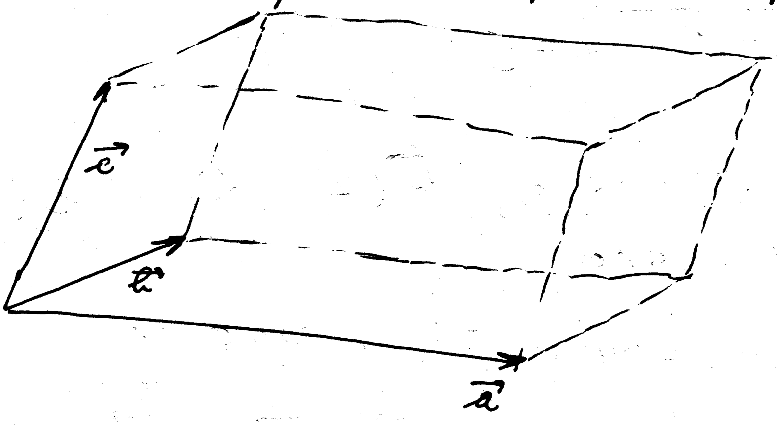
$y > 0 \Rightarrow \vec{r}(45, 24, 0)$

traženi vektor

Mješoviti proizvod tri vektora

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

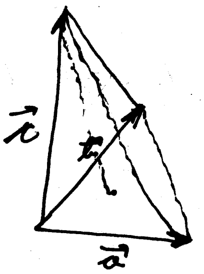
$(\vec{a}\times\vec{b})\cdot\vec{c}$ je broj koji je jednak zapremini paralelopipeda

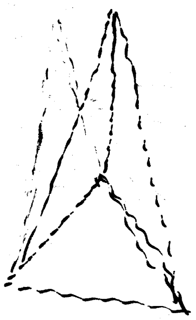


Ako je $(\vec{a}\times\vec{b})\cdot\vec{c} = 0$, tada su $\vec{a}$, $\vec{b}$ i $\vec{c}$ komplanarni vektori

Zapremina tetraedra (piramide) kojeg obrazuju vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$

$$V = \frac{1}{6} |(\vec{a} \times \vec{b}) \cdot \vec{c}|$$





① Provjeriti da li su vektori $\vec{a}(-1,3,2)$, $\vec{b}(2,-3,-4)$ i $\vec{c}(-3,12,6)$ komplanarni. Ako jesu izraziti vektor $\vec{c}$ preko vektora $\vec{a}$ i $\vec{b}$.

Rj: $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$ uslov komplanarnosti

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} -1 & 3 & 2 \\ 2 & -3 & -4 \\ -3 & 12 & 6 \end{vmatrix} \xrightarrow[III_k + I_k \cdot 2]{II_k + I_k \cdot (3)} \begin{vmatrix} -1 & 0 & 0 \\ 2 & 3 & 0 \\ -3 & 3 & 0 \end{vmatrix} = 0$$

vektori su komplanarni

$\vec{c} = \alpha \vec{a} + \beta \vec{b}$

$(-3, 12, 6) = \alpha(-1,3,2) + \beta(2,-3,-4)$

$$\begin{aligned} -\alpha + 2\beta &= -3 \\ 3\alpha - 3\beta &= 12 \quad /:3 \\ 2\alpha - 4\beta &= 6 \quad /\cdot(-2) \end{aligned}$$

$$\begin{aligned} -\alpha + 2\beta &= -3 \\ + \quad \alpha - \beta &= 4 \end{aligned}$$

$\beta = 1 \qquad \alpha = 5$

$\vec{c} = 5\vec{a} + \vec{b}$

vektor $\vec{c}$ razložen preko vektora $\vec{a}$ i $\vec{b}$

② Vektori $\vec{a}(1, 2\alpha, 1)$, $\vec{b}(2, \alpha, \alpha)$ i $\vec{c}(3\alpha, 2, -\alpha)$ su ivice tetraedra

a) Odrediti zapreminu tog tetraedra

b) Odrediti α tako da $\vec{a}$, $\vec{b}$ i $\vec{c}$ budu komplanarni i u tom slučaju izraziti vektor $\vec{a}$ preko vektora $\vec{b}$ i $\vec{c}$.

Rj. $\vec{a}(1, 2\alpha, 1)$

$\vec{b}(2, \alpha, \alpha)$

$\vec{c}(3\alpha, 2, -\alpha)$

a) $V=\frac{1}{6}\left|(\vec{a}\times\vec{b})\cdot\vec{c}\right|$

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} 1 & 2\alpha & 1 \\ 2 & \alpha & \alpha \\ 3\alpha & 2 & -\alpha \end{vmatrix} \overset{I_k - III_k}{\underset{II_k - III_k}{=}} \begin{vmatrix} 0 & 2\alpha-1 & 1 \\ 2-\alpha & 0 & \alpha \\ 4\alpha & 2+\alpha & -\alpha \end{vmatrix}$$

$$= -(2\alpha-1)\begin{vmatrix} 2-\alpha & \alpha \\ 4\alpha & -\alpha \end{vmatrix} + \begin{vmatrix} 2-\alpha & 0 \\ 4\alpha & 2+\alpha \end{vmatrix} =$$

$(2\alpha^2-\alpha)(3\alpha+2)$

$$= (1-2\alpha)\begin{vmatrix} 2+3\alpha & 0 \\ 4\alpha & -\alpha \end{vmatrix} + (4-\alpha^2) = (1-2\alpha)(-\alpha)\cdot(2+3\alpha)+4-\alpha^2 =$$

$$= 6\alpha^3 \underline{+4\alpha^2}\ \underline{-3\alpha^2} - 2\alpha + 4 \underline{-\alpha^2} = 2(3\alpha^3-\alpha+2)$$

$$V=\frac{1}{3}\left|3\alpha^3-\alpha+2\right|$$

zapremina tetraedra

b) $3\alpha^3-\alpha+2=0$

-1 je nula ovog polinoma pa

$(3\alpha^3-\alpha+2):(\alpha+1) = 3\alpha^2-3\alpha+2$

$-\ \underline{3\alpha^3+3\alpha^2}$

$-3\alpha^2-\alpha+2$

$-\ \underline{-3\alpha^2-3\alpha}$

$2\alpha+2$

$\underline{2\alpha+2}$

$=\ =$

tj. $3\alpha^3-\alpha+2=(\alpha+1)(3\alpha^2-3\alpha+2)$

$(\alpha+1)\underbrace{(3\alpha^2-3\alpha+2)}=0$

$D=9-24<0$

$a>0$

$3\alpha^2-3\alpha+2$ je uvijek pozitivno

$\Rightarrow \alpha=-1$

$\vec{a}(1,-2,1)$

$\vec{b}(2,-1,-1)$

$\vec{c}(-3,2,1)$

$\vec{a}=\lambda\vec{b}+\mu\vec{c}$

$(1,-2,1)=\lambda(2,-1,-1)+\mu(-3,2,1)$

$2\lambda-3\mu=1$

$-\lambda+2\mu=-2 \quad |\cdot 2$

$-\lambda+\mu=1 \quad |\cdot 2$

$2\lambda-3\mu=1 \quad (1)$

$-2\lambda+4\mu=-4 \quad (2)$

$-2\lambda+2\mu=2 \quad (3)$

(2)+(1): $\mu=-3 \Rightarrow \lambda=-4$

(3)+(1):

$\vec{a}=-4\vec{b}-3\vec{c}$

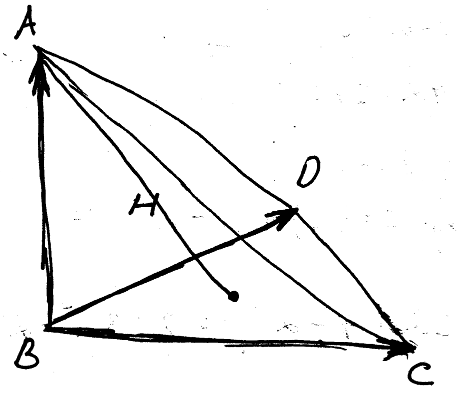
vektor $\vec{a}$ izražen preko $\vec{b}$ i $\vec{c}$

③ Date su tačke $A(3,2,1)$, $B(4,1,-2)$, $C(-5,-4,8)$ i $D(6,3,7)$. Odrediti:

a) zapreminu tetraedra ABCD.

b) visinu tetraedra koja odgovara osnovici BCD.

Rj.



$$\left.\begin{matrix} B(4,1,-2) \\ A(3,2,1) \end{matrix}\right\} \Rightarrow \overrightarrow{BA}(-1,1,3)$$

$$D(6,3,7) \Rightarrow \overrightarrow{BD}(2,2,9)$$

$$C(-5,-4,8) \Rightarrow \overrightarrow{BC}(-9,-5,10)$$

a) $$V = \frac{1}{6}\left|(\overrightarrow{BC} \times \overrightarrow{BD}) \cdot \overrightarrow{BA}\right| = \frac{1}{6}\left|\begin{vmatrix} -9 & -5 & 10 \\ 2 & 2 & 9 \\ -1 & 1 & 3 \end{vmatrix}\right| \overset{I_k + II_k}{\underset{III_k - II_k \cdot 3}{=}} \frac{1}{6}\left|\begin{vmatrix} -14 & -5 & 25 \\ 4 & 2 & 3 \\ 0 & 1 & 0 \end{vmatrix}\right|$$

$$= \frac{1}{6}\left|\begin{vmatrix} -14 & 25 \\ 4 & 3 \end{vmatrix}\right| = \frac{1}{6}\left|-42-100\right| = \frac{142}{6} = \frac{71}{3}$$

Zapremina tetraedra ABCD iznosi $\frac{71}{3}$.

b) Zapremina piramide $V = \frac{B \cdot H_{BCD}}{3}$

$$B = P_{\Delta ACD} = \frac{1}{2}\left|\overrightarrow{BC} \times \overrightarrow{BD}\right| = \frac{1}{2}\sqrt{4225 + 10\,201 + 64} = \frac{1}{2}\sqrt{9 \cdot 1610} = \frac{3}{2}\sqrt{1610}$$

$$\overrightarrow{BC} \times \overrightarrow{BD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -9 & -5 & 10 \\ 2 & 2 & 9 \end{vmatrix} = (-45-20)\vec{i} - (-81-20)\vec{j} + (-18+10)\vec{k}$$
$$= (-65, 101, -8)$$

$$\frac{71}{3} = \frac{\frac{3}{2}\sqrt{1610} \cdot H_{BCD}}{3} \qquad / \cdot 3 \cdot 2$$

$$3\sqrt{1610} \cdot H_{BCD} = 142$$

$$H_{BCD} = \frac{142}{3\sqrt{1610}}$$ je visina tetraedra koja odgovara osnovici BCD.

(#) Dati su vektori $\vec{a}\{\lambda, 3, 3\}$, $\vec{b}\{0, \lambda-1, \lambda+1\}$ i $\vec{c}\{\lambda, 3, 4\}$. Odrediti sve vrijednosti parametra λ tako da ovi vektori budu komplanarni, pa za veću vrijednost parametra λ razložiti vektor $\vec{a}$ preko vektora $\vec{b}$ i $\vec{c}$.

Rj.

Vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ su komplanarni akko $(\vec{a}\times\vec{b})\cdot\vec{c}=0$

$$(\vec{a}\times\vec{b})\cdot\vec{c}=\begin{vmatrix}\lambda & 3 & 3\\ 0 & \lambda-1 & \lambda+1\\ \lambda & 3 & 4\end{vmatrix} \underline{\underline{\ III_R-I_R\ }} \begin{vmatrix}\lambda & 3 & 3\\ 0 & \lambda-1 & \lambda+1\\ 0 & 0 & 1\end{vmatrix}=\begin{vmatrix}\lambda & 3\\ 0 & \lambda-1\end{vmatrix}=$$

$$=\lambda(\lambda-1)$$

$$\lambda(\lambda-1)=0$$

$$\lambda_1=0 \quad \lambda_2=1$$

Za vrijednost $\lambda=1$ vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ su komplanarni.

za $\lambda=1$ $\quad\vec{a}\{1, 3, 3\}$, $\vec{b}\{0, 0, 2\}$, $\vec{c}\{1, 3, 4\}$

$$\vec{a}=\alpha\vec{b}+\beta\vec{c}$$

$$\{1, 3, 3\}=\alpha\{0, 0, 2\}+\beta\{1, 3, 4\}$$

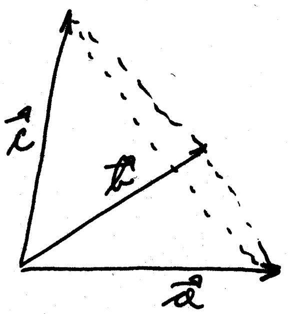
$$0\cdot\alpha+\beta=1$$
$$0\cdot\alpha+3\beta=3$$
$$2\alpha+4\beta=3$$
$$\overline{\beta=1}$$

$$2\alpha+4=3$$
$$2\alpha=-1$$
$$\alpha=-\frac{1}{2}$$

$\vec{a}=-\frac{1}{2}\vec{b}+\vec{c}$ vektor $\vec{a}$ razložen preko vektora $\vec{b}$ i $\vec{c}$

(#) Vektori $\vec{a}=(-1,-3,1)$, $\vec{b}=(\lambda,3,4)$ i $\vec{c}=(-5,-9,1)$ su ivice tetraedra. Odrediti parametar λ tako da zapremina tetraedra iznosi 8. Za vrijednost $\lambda=6$ provjeriti da li su vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ komplanarni, pa ako jesu izraziti vektor $\vec{a}$ preko vektora $\vec{b}$ i $\vec{c}$

R:



$$V=\frac{1}{6}\left|(\vec{a}\times\vec{b})\cdot\vec{c}\right| = \left|\frac{1}{6}\begin{vmatrix}-1&-3&1\\ \lambda&3&4\\ -5&-9&1\end{vmatrix}\right| \begin{array}{l} I_R-III_R \\ \hline\hline II_R-III_R\cdot 4\end{array}$$

$$=\left|\frac{1}{6}\begin{vmatrix}4&6&0\\ \lambda+20&39&0\\ -5&-9&1\end{vmatrix}\right| = \left|\frac{1}{6}\begin{vmatrix}4&6\\ \lambda+20&39\end{vmatrix}\right| =$$

$$=\left|\frac{1}{6}(156-6\lambda-120)\right| = \left|\frac{1}{6}(36-6\lambda)\right| = \left|\frac{1}{6}\cdot 6(6-\lambda)\right|$$

$V=|6-\lambda|$

$V=8$ $\Rightarrow$ $\lambda=-2$ Za $\lambda=-2$ zapremina tetraedra iznosi 8.

Za vrijednost $\lambda=6$ zapremina tetraedra je 0 pa su vektori $\vec{a}=(-1,-3,1)$, $\vec{b}=(6,3,4)$ i $\vec{c}=(-5,-9,1)$ komplanarni.

$$\vec{a}=\alpha\vec{b}+\beta\vec{c}$$

$$(-1,-3,1)=(6\alpha,3\alpha,4\alpha)+(-5\beta,-9\beta,\beta)$$

$$\begin{array}{l} 6\alpha-5\beta=-1 \\ 3\alpha-9\beta=-3 \quad /:3 \\ 4\alpha+\beta=1 \\ \hline 6\alpha-5\beta=-1 \\ \alpha-3\beta=-1 \\ 4\alpha+\beta=1 \\ \hline \end{array}$$

$$\alpha=3\beta-1$$
$$6\alpha-5\beta=-1$$
$$6(3\beta-1)-5\beta=-1$$
$$18\beta-6-5\beta=-1$$
$$13\beta=5$$
$$\beta=\frac{5}{13}$$

$$\alpha=\frac{15}{13}-\frac{13}{13}$$
$$\alpha=\frac{2}{13}$$

$\vec{a}=\frac{2}{13}\vec{b}+\frac{5}{13}\vec{c}$ vektor $\vec{a}$ izražen preko vektora $\vec{b}$ i $\vec{c}$.

Zadaci za vježbu:

① Kakav međusobni položaj zauzimaju vektori $\vec{a}$ i $\vec{b}$ ako je $|\vec{a}+\vec{b}|=|\vec{a}|+|\vec{b}|$.

② U trouglu ΔABC data je tačka D na stranici BC tako da je $\overline{BD}=\frac{1}{3}\overline{BC}$, a na duži $\overline{AD}$ data je tačka E tako da je duž $\overline{AE}=\frac{1}{4}\overline{AD}$. Izračunati koordinate tačke C ako se zna da je $A(2,0,1)$, $B(-1,1,4)$ i $E(1,3,2)$.

③ Dati su vektori $\vec{u}=6\vec{i}+\vec{j}+\vec{k}$, $\vec{v}=3\vec{j}-\vec{k}$ i $\vec{w}=-2\vec{i}+3\vec{j}+5\vec{k}$. Odrediti x tako da $\vec{u}+x\vec{v}\perp\vec{w}$.

⑯ Koliki ugao obrazuju vektori $\vec{a}$ i $\vec{b}$ ako su vektori $5\vec{a}-3\vec{b}\perp 2\vec{a}+4\vec{b}$ i ako je $|\vec{a}|=3$ i $|\vec{b}|=2$.

⑰ Dokazati da se prave na kojima leže visine trougla sijeku u istoj tački.

⑱ Odrediti visinu h_B spuštenu iz vrha B u trouglu ΔABC s vrhovima $A(1,-2,8)$, $B(0,0,4)$ i $C(6,2,0)$

⑲ Izračunati zapreminu paralelopipeda razapetog vektorima $\vec{a}=\vec{i}-3\vec{j}+k$, $\vec{b}=2\vec{i}+\vec{j}-3\vec{k}$ i $\vec{c}=\vec{i}+2\vec{j}+\vec{k}$.

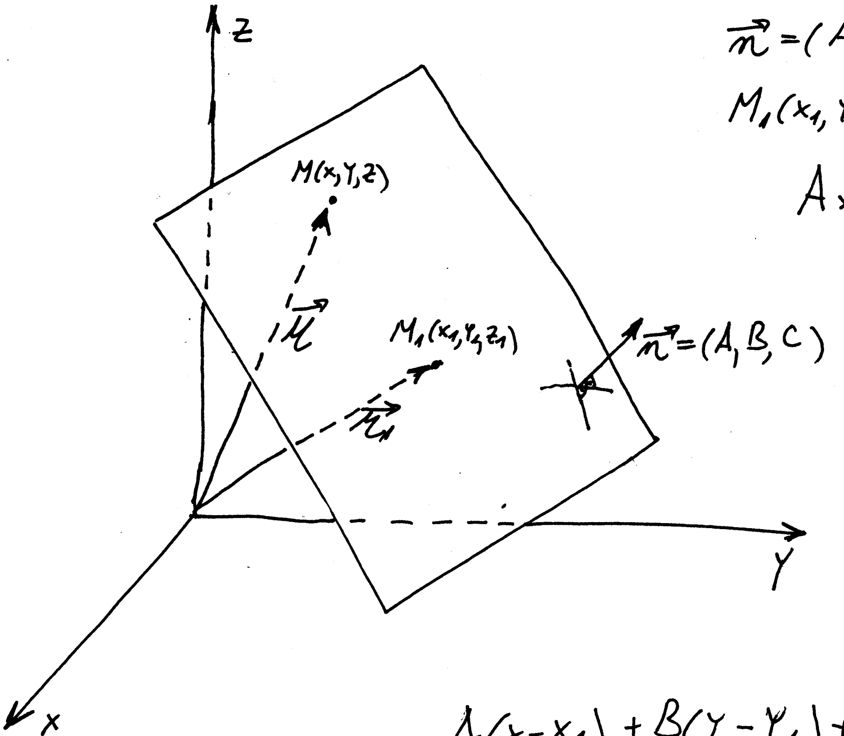
⑳ Izračunati visinu paralelopipeda razapetog vektorima $\vec{a}=3\vec{i}+2\vec{j}-5\vec{k}$, $\vec{b}=\vec{i}-\vec{j}+4\vec{k}$ i $\vec{c}=\vec{i}-3\vec{j}+\vec{k}$ ako je za osnovicu uzet paralelogram razapet vektorima $\vec{a}$ i $\vec{b}$.

㉑ Odredite α tako da zapremina tetraedra razapetog vektorima $\vec{a}$, $\vec{b}$ i $\alpha\vec{c}$ iznosi $\frac{2}{3}$, gdje je $\vec{a}=\vec{i}+\vec{j}-2\vec{k}$, $\vec{b}=2\vec{i}+\vec{j}-\vec{k}$ i $\vec{c}=\vec{i}-\frac{1}{3}\vec{k}$.

㉒ Zadan je trokut s vrhovima $A(2,3,2)$, $B(0,1,1)$ i $C(4,4,0)$. Odredite koordinate tačke S presjeka simetrale unutrašnjeg ugla pri vrhu A i simetrale stranice AB.

㉓ Dokažite vektorskim računom da se u trouglu simetrale stranica sijeku u jednoj tački.

Ravan



$\vec{n}=(A,B,C)$ vektor normale

$M_1(x_1,y_1,z_1)$ tačka u ravni

$$Ax+By+Cz+D=0$$

opšti oblik jednačine ravni

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

segmentni oblik jednačine ravni ($(a,0,0)$, $(0,b,0)$ i $(0,0,c)$ su tačke presjeka ravni sa x, y i z-osom)

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

skalarni oblik jednačine ravni kroz tačku $M_1(x_1,y_1,z_1)$

$(\vec{r}-\vec{r_1})\cdot\vec{n}=0$ vektorski oblik jednačine ravni

$$d=\frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}$$

rastojanje tačke $M_1(x_1,y_1,z_1)$ od ravni $Ax+By+Cz+D=0$.

Ako su date dvije ravni

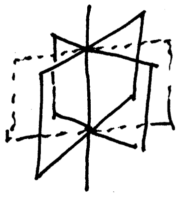
$$A_1x+B_1y+C_1z+D_1=0$$
$$A_2x+B_2y+C_2z+D_2=0$$

$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}$ uslov paralelnosti dvije ravni ($\vec{n_1}$ i $\vec{n_2}$ kolinearni)

$A_1A_2+B_1B_2+C_1C_2=0$ $\Rightarrow$ ravni međusobno normalne

$\cos\varphi=\pm\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$ ugao između dvije ravni

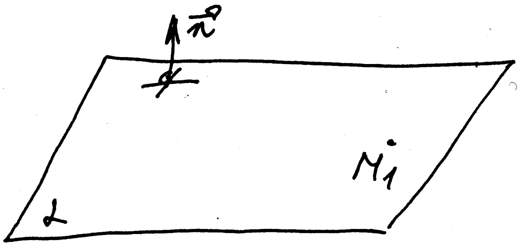
$A_1x+B_1y+C_1z+D_1+\lambda(A_2x+B_2y+C_2z+D_2)=0$ pramen ravni (skup svih ravni koje prolaze kroz istu pravu)



λ LAMBDA

(#) Napisati jednačinu ravni koja sadrži tačku $M_1(-2,1,3)$ i normalna je na vektor $\vec{n}=(1,2,7)$.

Rj:



α: ?

$M_1(-2,1,3)$

$\vec{n}=(1,2,7)$

$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$ jednačina tražene ravni

$A=1,\ B=2,\ C=7$

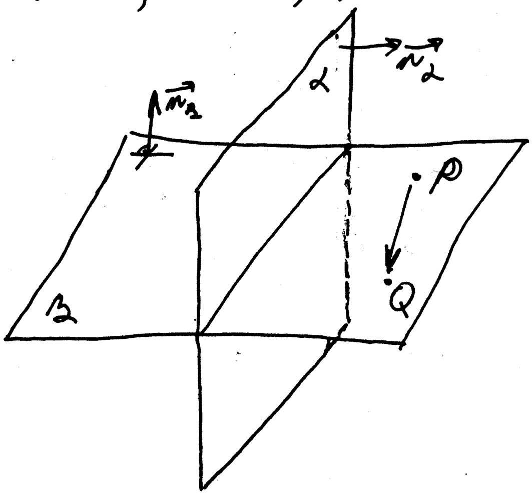
$$1(x+2)+2(y-1)+7(z-3)=0$$

$$x+2+2y-2+7z-21=0$$

$$x+2y+7z-21=0$$ jednačina tražene ravni

(#) Napisati jednačinu ravni koja prolazi kroz tačke $P(1,1,1)$, $Q(0,1,-1)$ i normalna je na ravan $\alpha: x+y+z-1=0$.

Rj:



β: ?

$$\left.\begin{array}{l}\vec{n_\beta}\perp\vec{PQ}\\ \vec{n_\beta}\perp\vec{n_\alpha}\end{array}\right\} \Rightarrow \vec{n_\beta}\parallel\vec{n_\alpha}\times\vec{PQ}$$

$$\Downarrow$$

$$\exists k\in R:\ \vec{n_\beta}=k(\vec{n_\alpha}\times\vec{PQ})$$

$$\left.\begin{array}{l}P(1,1,1)\\ Q(0,1,-1)\end{array}\right. \Rightarrow \vec{PQ}=(-1,0,-2)$$

$$\vec{n_\alpha}=(1,1,1)$$

$$\vec{n_\alpha}\times\vec{PQ}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 1 & 1\\ -1 & 0 & -2\end{vmatrix}=\vec{i}(-2-0)-\vec{j}(-2+1)+\vec{k}(0+1)=-2\vec{i}+\vec{j}+\vec{k}$$

$$=(-2,1,1)$$

$\Rightarrow \vec{n_\beta}=k(-2,1,1)$ gdje je k neki realan broj, $k\neq 0$

$n_\beta=(-2k,k,k)$

$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$ $\quad P(\overset{x_1}{1},\overset{y_1}{1},\overset{z_1}{1})$

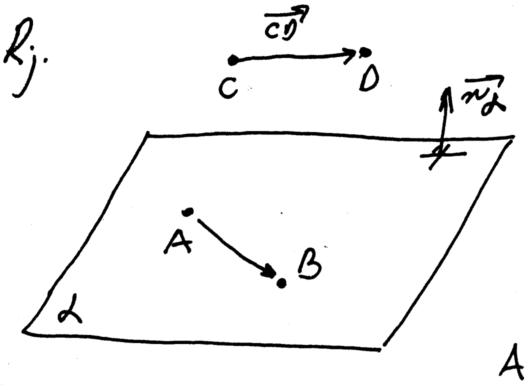
$$-2k(x-1)+k(y-1)+k(z-1)=0 \quad /:k$$

$$-2x+2+y-1+z-1=0$$

$$-2x+y+z=0$$ jednačina tražene ravni

(#) Date su tačke $A(0,3,4)$, $B(-1,2,3)$, $C(1,-2,-1)$ i $D(4,-1,1)$. Napisati jednačinu ravni koja sadrži tačke A i B, i paralelna je sa vektorom $\vec{CD}$.

Rj.



$\alpha: ?$ $\quad A(x-x_1)+B(y-y_1)+C(z-z_1)=0$ jednačina ravni

$$\left.\begin{array}{l}\vec{n_\alpha}\perp\vec{AB}\\ \vec{n_\alpha}\perp\vec{CD}\end{array}\right\} \Rightarrow \vec{n_\alpha}\parallel\vec{AB}\times\vec{CD}$$

$$\Downarrow$$

$$\exists k\in\mathbb{R}\ \ \vec{n_\alpha}=k(\vec{AB}\times\vec{CD})$$

$$\left.\begin{array}{l}A(0,3,4)\\ B(-1,2,3)\end{array}\right. \Rightarrow \vec{AB}=(-1,-1,-1)$$

$$\left.\begin{array}{l}C(1,-2,-1)\\ D(4,-1,1)\end{array}\right. \Rightarrow \vec{CD}=(3,1,2)$$

$$\vec{AB}\times\vec{CD}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\-1&-1&-1\\3&1&2\end{vmatrix}=-\vec{i}-\vec{j}+2\vec{k}$$

$$=(-1,-1,2)$$

$$B(\overset{x_1}{-1},\overset{y_1}{2},\overset{z_1}{3})$$

$\Downarrow$ $\vec{n_\alpha}=k(-1,-1,2)$, gdje je k neki realan broj.

$=-k(1,1,-2)$

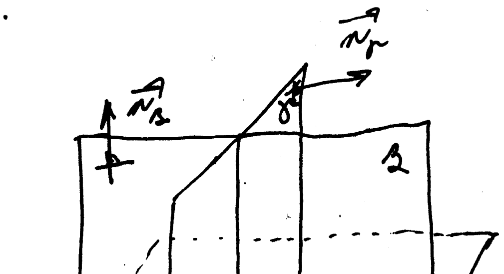
$$-k\cdot1(x+1)-k\cdot1\cdot(y-2)-k\cdot(-2)(z-3)=0 \quad /:(-k),\ k\neq0$$

$$x\underline{\underline{+1}}+y\underline{\underline{-2}}-2z\underline{\underline{+6}}=0$$

$x+y-2z+5=0$ jednačina tražene ravni

(#) Napisati jednačinu ravni koja prolazi kroz tačku $M_1(2,0,-1)$ i normalna je na ra ravnima $2x-y-3=0$ i $x+y-z+1=0$.

Rj.



$\alpha: ?$

$\beta: 2x-y-3=0,\quad \vec{n_\beta}=(2,-1,0)$

$\gamma: x+y-z+1=0,\quad \vec{n_\gamma}=(1,1,-1)$

$$\left.\begin{array}{c}\text{Ako } M_1 \text{ uvrstim u } \beta \text{ imam}\\ 2\cdot2-0-3\neq0\\ \text{Ako } M_1 \text{ uvrstim u } \gamma \text{ imam}\\ 2+0+1+1\neq0\end{array}\right\}\Rightarrow \begin{array}{l}M_1\notin\beta\\ \text{i } M_1\notin\gamma\end{array}$$

$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$

jednačina tražene ravni

$$\left.\begin{array}{l}\vec{n_\alpha}\perp\vec{n_\beta}\\ \vec{n_\alpha}\perp\vec{n_\gamma}\end{array}\right\} \Rightarrow \vec{n_\alpha}\parallel \vec{n_\beta}\times\vec{n_\gamma}$$

$$\Downarrow$$

$$\exists k\in\mathbb{R}:\ \vec{n_\alpha}=k(\vec{n_\beta}\times\vec{n_\gamma})$$

$$\vec{n_\beta}\times\vec{n_\gamma}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -1 & 0\\ 1 & 1 & -1\end{vmatrix}=\vec{i}(1-0)-\vec{j}(-2-0)+\vec{k}(2+1)=\vec{i}+2\vec{j}+3\vec{k}$$

$$=(1,2,3)$$

$$\vec{n_\alpha}=k(1,2,3)=(\overset{A}{k},\overset{B}{2k},\overset{C}{3k})$$ gdje je k neki realan broj, $k\neq 0$

$$k(x-2)+2k(y-0)+3k(z+1)=0 \quad /:k$$

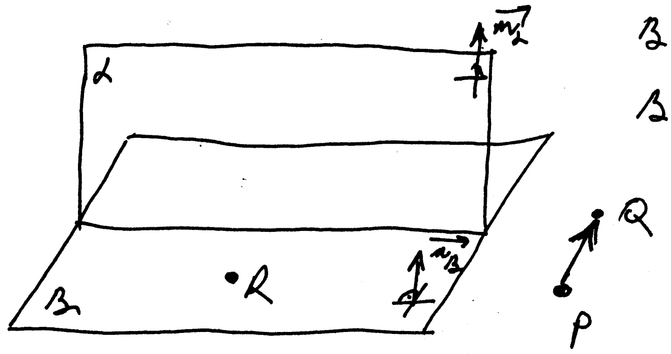
$$x+2y+3z-2+3=0$$

$$x+2y+3z+1=0$$

jednačina tražene ravni

(#) Date su tačke $P(1,1,-1)$, $Q(1,2,0)$ i $R(-1,0,0)$. Napisati jednačinu ravni koja je normalna na ravan α: $2x-y+5z-3=0$, koja je paralelna sa vektorom $\vec{PQ}$ i sadrži tačku R.

Rj.



β: ?

β: $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$

$$\begin{array}{l}P(1,1,-1)\\ Q(1,2,0)\end{array} \Rightarrow \vec{PQ}=(0,1,1)$$

$$\vec{n_\alpha}=(2,-1,5)$$

$$\left.\begin{array}{l}\vec{n_\beta}\perp\vec{n_\alpha}\\ \vec{n_\beta}\perp\vec{PQ}\end{array}\right\} \Rightarrow \vec{n_\beta}\parallel \vec{n_\alpha}\times\vec{PQ}$$

$$\Downarrow$$

$$\exists k\in\mathbb{R}:\ \vec{n_\beta}=k(\vec{n_\alpha}\times\vec{PQ})$$

$$\vec{n_\alpha}\times\vec{PQ}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -1 & 5\\ 0 & 1 & 1\end{vmatrix}=-6\vec{i}-2\vec{j}+2\vec{k}=(-6,-2,2)$$

$$\Downarrow$$

$$\vec{n_\beta}=k(-6,-2,2)=-2k(3,1,-1)$$

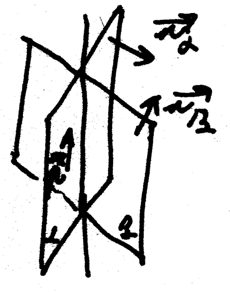
$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

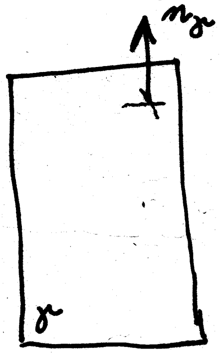
$$-2k\cdot 3(x+1)-2k\cdot 1(y-0)-2k(-1)(z-0)=0 \quad /:(-2k)$$

$$3x+y+z+3=0$$ jednačina tražene ravni

Kroz presjek ravni $4x-y+3z-1=0$ i $x+5y-z+2=0$ postaviti ravan koja je normalna na ravan $2x-y+5z-3=0$.

Rj. I način:





$\alpha: 4x-y+3z-1=0$

$\beta: x+5y-z+2=0$

$\gamma: 2x-y+5z-3=0$

$\vec{n_\alpha}=(4,-1,3)$

$\vec{n_\beta}=(1,5,-1)$

$\vec{n_\gamma}=(2,-1,5)$

$$\left.\begin{array}{l}\vec{p}\perp\vec{n_\alpha}\\ \vec{p}\perp\vec{n_\beta}\end{array}\right\} \Rightarrow \vec{p}\parallel\vec{n_\alpha}\times\vec{n_\beta}$$

$$\Downarrow$$

$$\vec{p}=k\cdot(\vec{n_\alpha}\times\vec{n_\beta})$$
$k\in\mathbb{R}$

$$\vec{n_\alpha}\times\vec{n_\beta}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\4&-1&3\\1&5&-1\end{vmatrix}=(1-15)\vec{i}-(-4-3)\vec{j}+(20+1)\vec{k}=(-14,7,21)$$

pa za $\vec{p}$ mogu uzeti $\vec{p}=(-2,1,3)$

$$\left.\begin{array}{l}\vec{n}\perp\vec{p}\\ \vec{n}\perp\vec{n_\gamma}\end{array}\right\} \Rightarrow \vec{n}\parallel\vec{p}\times\vec{n_\gamma},\qquad \vec{p}\times\vec{n_\gamma}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\-2&1&3\\2&-1&5\end{vmatrix}=(8,16,0)\Rightarrow$$

$$\Rightarrow \vec{n}=(1,2,0)$$

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$ jednačina ravni kroz tačku (x_1,y_1,z_1) i vektor normale $\vec{n}=(A,B,C)$.

nađimo tačku koja pripada presjeku ravni $\alpha\cap\beta$

$$\begin{array}{l}4x-y+3z-1=0\\ \underline{x+5y-z+2=0\ /\cdot 3}\end{array}\qquad \begin{array}{l}4x-y+3z-1=0\\ \underline{+\ 3x+15y-3z+6=0}\end{array}\qquad 7x+14y+5=0$$

$$x=\frac{2}{7}\Rightarrow 14y=-2-5$$
$$y=-\frac{1}{2}$$

$$4\cdot\frac{2}{7}+\frac{1}{2}+3z-1=0\Rightarrow 3z=-\frac{8}{7}-\frac{1}{2}+1=\frac{1}{2}-\frac{8}{7}=\frac{7-16}{14}=\frac{-9}{14}\ /\cdot\frac{1}{3}$$

$$\Rightarrow z=-\frac{3}{14}$$

$$M\left(\frac{2}{7},-\frac{1}{2},-\frac{3}{14}\right)$$

$$1\cdot\left(x-\frac{2}{7}\right)+2\left(y+\frac{1}{2}\right)+0\cdot\left(z+\frac{3}{14}\right)=0$$

$$x-\frac{2}{7}+2y+1=0\Longrightarrow 7x+14y+5=0$$ jednačina tražene ravni

II način: koristimo formulu pramena pravih

$$4x-y+3z-1+\lambda(x+5y-z+2)=0$$

$$(4+\lambda)x+(-1+5\lambda)y+(3-\lambda)z-1+2\lambda=0$$

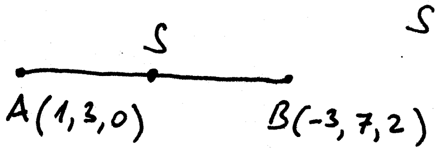
$$\vec{n}=(4+\lambda,-1+5\lambda,3-\lambda)$$

$$\vec{n}\perp\vec{n_\gamma}=\vec{n}\cdot\vec{n_\gamma}=0\Rightarrow\lambda=3$$

$$\Rightarrow 7x+14y+5=0$$ jednačina tražene ravni

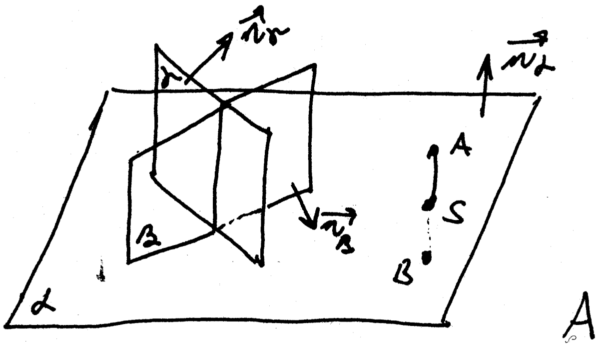
(#) Kroz središte S duži određene tačkama $A(1,3,0)$ i $B(-3,7,2)$ postaviti ravan α koja će biti okomita na ravan $\beta: 6x-4y+z=16$ i $\gamma: y+2z+1=0$. (Obavezno nacrtati sliku).

Rj:



$$S\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

$$S(-1, 5, 1)$$



$\beta: 6x-4y+z=16$

$\vec{n_\beta}=(6,-4,1)$

$\gamma: y+2z+1=0$

$\vec{n_\gamma}=(0,1,2)$

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

jednačina ravni kroz jednu tačku

$\vec{n_\alpha}=(A,B,C)$

$$\left.\begin{matrix}\vec{n_\alpha}\perp\vec{n_\beta}\\ \vec{n_\alpha}\perp\vec{n_\gamma}\end{matrix}\right\} \Rightarrow \vec{n_\alpha}\parallel\vec{n_\beta}\times\vec{n_\gamma}$$

$$\Downarrow$$

$$\exists k\in R \quad \vec{n_\alpha}=k(\vec{n_\beta}\times\vec{n_\gamma})$$

$$\vec{n_\beta}\times\vec{n_\gamma}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 6 & -4 & 1\\ 0 & 1 & 2\end{vmatrix}=-9\vec{i}-12\vec{j}+6k=(-9,-12,6)$$

pa za $\vec{n_\alpha}$ možemo uzeti

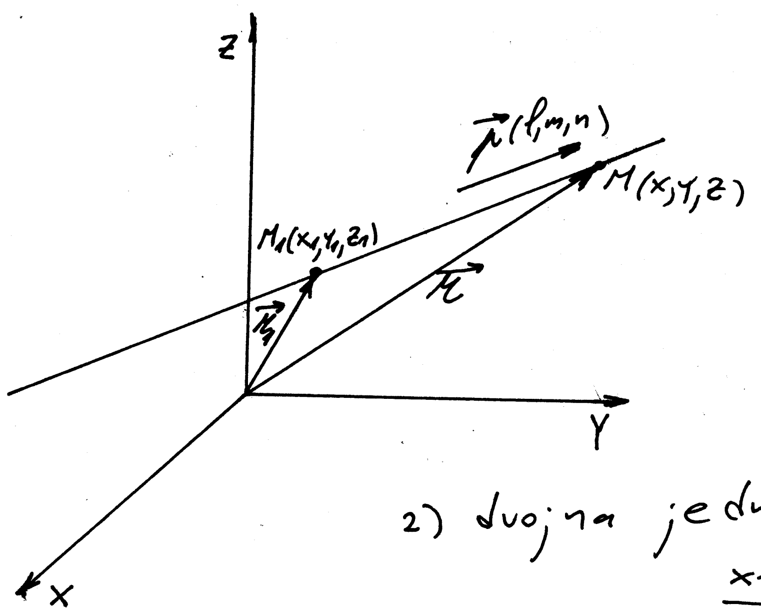
$\vec{n_\alpha}=(3,4,-2)$

$$3(x-(-1))+4(y-5)+(-2)(z-1)=0$$

$$3x+4y-2z+3-20+2=0$$

$$3x+4y-2z-15=0$$ jednačina tražene ravni

Prava u prostoru



Prava koja prolazi kroz tačku $M_1(x_1,y_1,z_1)$ i koja ima vektor pravca $\vec{p}=(l,m,n)$ ima sljedeće jednačine:

1) vektorska jednačina
$$(\vec{r}-\vec{r_1})\times\vec{p}=0$$

2) dvojna jednačina u kanoničkom obliku
$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

3) parametarske jednačine
$$x=x_1+lt$$
$$y=y_1+mt$$
$$z=z_1+nt$$

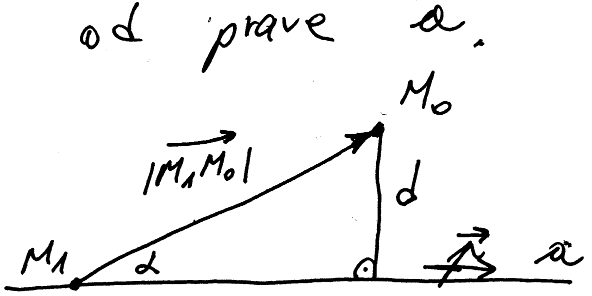
$$\begin{cases} A_1x+B_1y+C_1z+D_1=0 \\ A_2x+B_2y+C_2z+D_2=0 \end{cases}$$ jednačina prave koja je data presjekom dvije ravni

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$ jednačina ravni kroz dvije tačke

Potreban uslov da se prave $a: \frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ i $b: \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ sijeku jest da je
$$\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

$$d=\frac{|(\vec{p_1}\times\vec{p_2})\cdot\overrightarrow{M_1M_2}|}{|\vec{p_1}\times\vec{p_2}|}$$ udaljenost između dvije prave

Izvesti formulu za rastojanje tačke $M_0 \notin a$ od prave a.



$M_1 \in a$

Prava a ima vektor pravca $\vec{p}$

$$\sin\alpha=\frac{d}{|\overrightarrow{M_1M_0}|} \Rightarrow d=|\overrightarrow{M_1M_0}|\cdot\sin\alpha$$

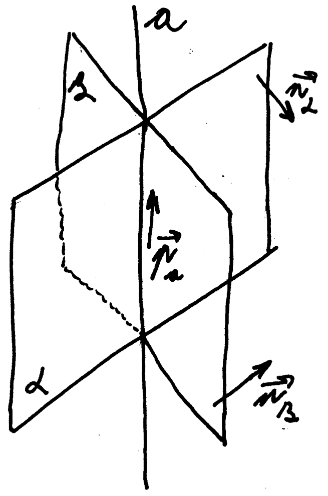
Od ranije znamo da je $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\cdot\sin\measuredangle(\vec{a},\vec{b})$

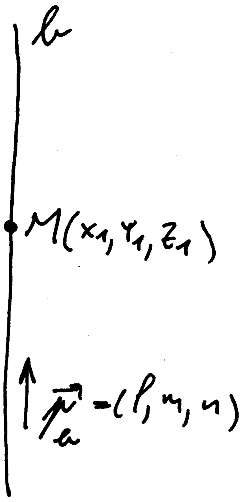
pa ćemo imati $\sin(\vec{p},\overrightarrow{M_1M_0})=\dfrac{|\vec{p}\times\overrightarrow{M_1M_0}|}{|\vec{p}|\cdot|\overrightarrow{M_1M_0}|}$

dobićemo $d=\dfrac{|\vec{p}\times\overrightarrow{M_1M_0}|}{|\vec{p}|}$ rastojanje tačke M_0 od prave a

Naći jednačinu prave koja sadrži tačku $M(-4,3,0)$ i paralelna je pravoj $\begin{cases} x-2y+z-4=0 \\ 2x+y-z=0 \end{cases}$.

Rj.





$$b:\ \frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

$\alpha:\ x-2y+z-4=0$

$\beta:\ 2x+y-z=0$

$\vec{n}_\alpha=(1,-2,1)$

$\vec{n}_\beta=(2,1,-1)$

$$\left.\begin{matrix}\vec{p}_a\perp\vec{n}_\alpha\\ \vec{p}_a\perp\vec{n}_\beta\end{matrix}\right\}\Rightarrow \vec{p}_a\parallel\vec{n}_\alpha\times\vec{n}_\beta$$

$$\Downarrow$$

$$\vec{p}_a=k(\vec{n}_\alpha\times\vec{n}_\beta)$$

$k\in\mathbb{R}$

$\vec{p}_a\parallel\vec{p}_b \Rightarrow \vec{p}_b=t\cdot\vec{p}_a$

$t\in\mathbb{R}$

$$\vec{n}_\alpha\times\vec{n}_\beta=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&-2&1\\2&1&-1\end{vmatrix}=\vec{i}\,(2-1)-\vec{j}\,(-1-2)+\vec{k}\,(1+4)=$$

$$=\vec{i}+3\vec{j}+5\vec{k}=(1,3,5)$$

$\vec{p}_b=(1,3,5)$

$M(-4,3,0)$

$\dfrac{x+4}{1}=\dfrac{y-3}{3}=\dfrac{z}{5}$ jednačina prave koja sadrži tačku M i paralelna je pravoj

(#) Odrediti λ u jednačini prave $\frac{x-3}{1} = \frac{Y-1}{\lambda} = \frac{z+2}{1}$ da bi se sjekla sa pravom $\frac{x-1}{2} = \frac{Y-2}{1} = \frac{z-1}{-1}$ i u tom slučaju naći presječnu tačku i ugao između pravih.

Rj:

$a: \frac{x-3}{1} = \frac{Y-1}{\lambda} = \frac{z+2}{1}, \quad \vec{p}_a = (1, \lambda, 1), \quad x_1 = 3, \quad Y_1 = 1, \quad z_1 = -2$

$b: \frac{x-1}{2} = \frac{Y-2}{1} = \frac{z-1}{-1}, \quad \vec{p}_b = (2, 1, -1), \quad x_2 = 1, \quad Y_2 = 2, \quad z_2 = 1$

Potreban uslov da se prave sijeku: $\begin{vmatrix} x_2-x_1 & Y_2-Y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$.

$$\begin{vmatrix} -2 & 1 & 3 \\ 1 & \lambda & 1 \\ 2 & 1 & -1 \end{vmatrix} \frac{I_R + III_R \cdot 3}{II_R + III_R} \begin{vmatrix} 4 & 4 & 0 \\ 3 & \lambda+1 & 0 \\ 2 & 1 & -1 \end{vmatrix} = (-1) \begin{vmatrix} 4 & 4 \\ 3 & \lambda+1 \end{vmatrix} = (-1)(4\lambda + 4 - 12) = (-1)(4\lambda - 8)$$

$(-1)(4\lambda - 8) = 0$

$\lambda = 2$

Za vrijednost $\lambda = 2$ prave a i b se sijeku.

a:

$\frac{x-3}{1} = \frac{Y-1}{2} = \frac{z+2}{1} \quad (=t)$

$x - 3 = t \qquad x = t + 3$

$Y - 1 = 2t \qquad Y = 2t + 1$

$z + 2 = t \qquad z = t - 2$

b:

$\frac{x-1}{2} = \frac{Y-2}{1} = \frac{z-1}{-1} \quad (=s)$

$x - 1 = 2s \qquad x = 2s + 1$

$Y - 2 = s \qquad Y = s + 2$

$z - 1 = -s \qquad z = -s + 1$

$t + 3 = 2s + 1$

$2t + 1 = s + 2$

$t - 2 = -s + 1$

$t - 2s = -2 \quad | \cdot 2$

$2t - s = 1$

$t + s = 3 \quad | \cdot 2$

$2t - 4s = -4 \quad (1)$

$2t - s = 1 \quad (2)$

$2t + 2s = 6 \quad (3)$

$(1)-(3): \; -6s = -10$

$(2)-(3): \; -3s = -5$

$s = \frac{5}{3}$

$t = 2s - 2 = \frac{10}{3} - \frac{6}{3} = \frac{4}{3} \qquad x = \frac{4}{3} + 3 = \frac{13}{3}, \quad Y = \frac{8}{3} + 1 = \frac{11}{3}, \quad z = \frac{4}{3} - 2 = -\frac{2}{3}$

Presječna tačka pravih je $M\left(\frac{13}{3}, \frac{11}{3}, -\frac{2}{3}\right)$.

$\vec{p}_a \cdot \vec{p}_b = (1, 2, 1) \cdot (2, 1, -1) = 2 + 2 - 1 = 3$

$|\vec{p}_a| = \sqrt{1+4+1} = \sqrt{6}, \quad |\vec{p}_b| = \sqrt{4+1+1} = \sqrt{6}$

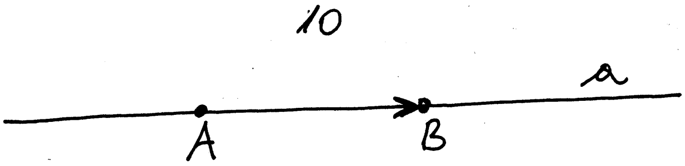
$\vec{p}_a \cdot \vec{p}_b = |\vec{p}_a| \cdot |\vec{p}_b| \cdot \cos\measuredangle(\vec{p}_a, \vec{p}_b)$

$\Rightarrow \cos\measuredangle(\vec{p}_a, \vec{p}_b) = \frac{\vec{p}_a \cdot \vec{p}_a}{|\vec{p}_a| \cdot |\vec{p}_a|} = \frac{3}{6} = \frac{1}{2} \Rightarrow \measuredangle(\vec{p}_a, \vec{p}_a) = 60°$ ugao između pravih

(#) Na pravoj $\frac{x-8}{8}=\frac{y-2}{-6}=\frac{z}{0}$ naći tačku čije rastojanje od tačke $A(8,2,0)$ iznosi 10.

Rj. $a: \frac{x-8}{8}=\frac{y-2}{-6}=\frac{z}{0} \quad (=t)$ $\qquad A(8,2,0)$

Tačka A pripada pravoj a.



$$a: \begin{cases} x=8t+8 \\ y=-6t+2 \\ z=0 \end{cases}$$

$|\overrightarrow{AB}|=\sqrt{64t^2+36t^2}$

$|\overrightarrow{AB}|=10$

Tražimo tačku B tako da je $|\overrightarrow{AB}|=10$

$\sqrt{100t^2}=10$

$10|t|=10$

$B(8t+8,-6t+2,0)$

$|t|=1$

$\overrightarrow{AB}=(8t,-6t,0)$

$t_1=-1 \quad t_2=1$

$B_1(0,8,0)$
$B_2(16,-4,0)$

Tačke B_1 i B_2 su tražene tačke

(#)✓ Naći rastojanje između ravni $\alpha: x-2y+z-1=0$ i ravni $\beta: 2x-4y+2z+1=0$.

(#)✓ Napisati jednačinu ravni koja prolazi kroz tačke $P(1,1,1)$, $Q(0,1,-1)$ i normalna je na ravan α: $x+y+z-1=0$.

(#)✓ Odrediti jednačinu ravni koja je paralelna sa vektorima $\overrightarrow{PQ}$ i $\overrightarrow{RT}$ i prolazi kroz tačku $M(0,1,0)$ ako su $P(-3,-2,-2)$, $Q(0,0,2)$, $R(-3,1,0)$ i $T(1,2,2)$.

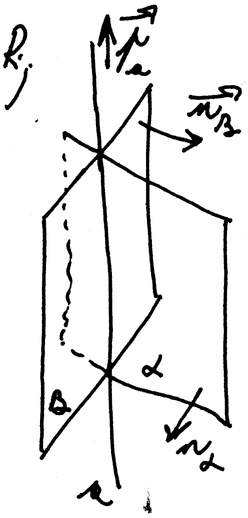
(#)✓ Odrediti uglove kojeg obrazuju prava $a: \begin{cases} 2x-2y-z-8=0 \\ x+2y-2z-4=0 \end{cases}$ i prava $b: \begin{cases} 4x+y+3z-4=0 \\ 2x+2y-3z-11=0 \end{cases}$.

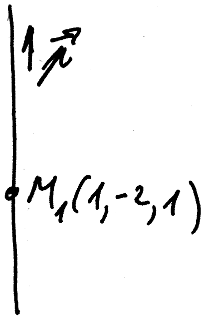
Rj. $\cos\varphi=\frac{4}{21}$

(#)✓ Odrediti presječnu tačku pravih $\begin{cases} 5x-2y+5z+3=0 \\ x+3y-4z-10=0 \end{cases}$ i $\begin{cases} 3x+10y-z-47=0 \\ 6x-2y+7z+3=0 \end{cases}$ Rj. $(2,4,-1)$

④ Kroz tačku $M_1(1,-2,1)$ povući pravu paralelnu pravoj $\begin{cases} x-y+z-4=0 \\ 2x+y-2z+5=0 \end{cases}$.

Rj:





$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$ jednačina prave kroz tačku $M(x_1,y_1,z_1)$

$\alpha: x-y+z-4=0$

$\vec{n}_\alpha=(1,-1,1)$ vektor normale na ravan α

$\beta: 2x+y-2z+5=0$

$\vec{n}_\beta=(2,1,-2)$ vektor normale na ravan β

$\vec{p}_a \parallel \vec{p}$

$$\left.\begin{matrix} \vec{p}_a \perp \vec{n}_\alpha \\ \vec{p}_a \perp \vec{n}_\beta \end{matrix}\right\} \Rightarrow \left.\begin{matrix} \vec{p}_a \parallel \vec{n}_\alpha \times \vec{n}_\beta \\ \vec{p} \parallel \vec{p}_a \end{matrix}\right\} \Rightarrow \vec{p} \parallel \vec{n}_\alpha \times \vec{n}_\beta$$

$$\vec{n}_\alpha \times \vec{n}_\beta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = (2-1)\vec{i}-(-2-2)\vec{j}+(1+2)\vec{k}=(1,4,3)$$

Za vektor pravca tražene prave mogu uzeti

$\vec{p}=(1,4,3)$

$M_1(\overset{x_1}{1}, \overset{y_1}{-2}, \overset{z_1}{1})$

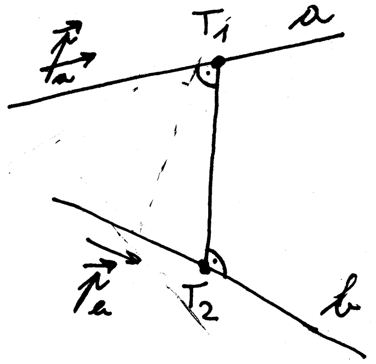
$$\frac{x-1}{1}=\frac{y+2}{4}=\frac{z-1}{3}$$

jednačina tražene prave

(#) Izračunati rastojanje između pravih

$$\frac{x-1}{4}=\frac{y}{-3}=\frac{z+5}{-1} \quad i \quad \frac{x}{-3}=\frac{y+4}{2}=\frac{z-1}{6}.$$

Rj.



I način:

$a: \quad \frac{x-1}{4}=\frac{y}{-3}=\frac{z+5}{-1}=s$

$$\left.\begin{array}{l} x-1=4s \\ y=-3s \\ z+5=-s \end{array}\right\} \Rightarrow \begin{array}{l} x=4s+1 \\ y=-3s \\ z=-s-5 \end{array}$$

$b: \quad \frac{x}{-3}=\frac{y+4}{2}=\frac{z-1}{6}=t$

$$\left.\begin{array}{l} x=-3t \\ y+4=2t \\ z-1=6t \end{array}\right\} \Rightarrow \begin{array}{l} x=-3t \\ y=2t-4 \\ z=6t+1 \end{array}$$

$$\left.\begin{array}{l} \overrightarrow{T_1T_2} \perp \vec{p}_a \\ \overrightarrow{T_1T_2} \perp \vec{p}_b \end{array}\right\} \Rightarrow \begin{array}{l} \overrightarrow{T_1T_2}\cdot\vec{p}_a=0 \\ \overrightarrow{T_1T_2}\cdot\vec{p}_b=0 \end{array}$$

$$\begin{array}{l} T_1(4s+1,\ -3s,\ -s-5) \\ T_2(-3t,\ 2t-4,\ 6t+1) \end{array} \Rightarrow \overrightarrow{T_1T_2}=(-3t-4s-1,\ 2t+3s-4,\ 6t+s+6)$$

$$d=|\overrightarrow{T_1T_2}|$$

$\vec{p}_a=(4,-3,-1)$

$\vec{p}_b=(-3,2,6)$

$$\vec{p}_a\times\vec{p}_b=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -3 & -1 \\ -3 & 2 & 6 \end{vmatrix}=\vec{i}(-16)-\vec{j}(21)+\vec{k}(-1)$$

$$=-16\vec{i}-21\vec{j}-\vec{k}$$

$\vec{p}_a\times\vec{p}_b=(-16,-21,-1)$

$$\left.\begin{array}{l} \vec{p}_a\cdot\overrightarrow{T_1T_2}=0 \\ \vec{p}_b\cdot\overrightarrow{T_1T_2}=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} -12s-13t+1=0 \\ 49s+24t+31=0 \end{array}\right\} \Rightarrow \begin{array}{l} s=\frac{-427}{349} \\ t=\frac{421}{349} \end{array}$$

$$\overrightarrow{T_1T_2}=\left(\frac{-752}{349},\ \frac{-987}{349},\ \frac{-47}{349}\right)=\left(\frac{-2^4\cdot 47}{349},\ \frac{-3\cdot 7\cdot 47}{349},\ \frac{-47}{349}\right)$$

$$d=|\overrightarrow{T_1T_2}|=\sqrt{\frac{4418}{349}}=\frac{94}{\sqrt{2\cdot 349}}=\frac{94}{\sqrt{698}}$$ rastojanje između pravih

II način

$|(\vec{p}_a\times\vec{p}_b)\cdot\overrightarrow{M_1M_2}|$ zapremina paralelopipeda $=V$

$|\vec{p}_a\times\vec{p}_b|$ površina paralelograma $=B$

$$V=B\cdot H$$

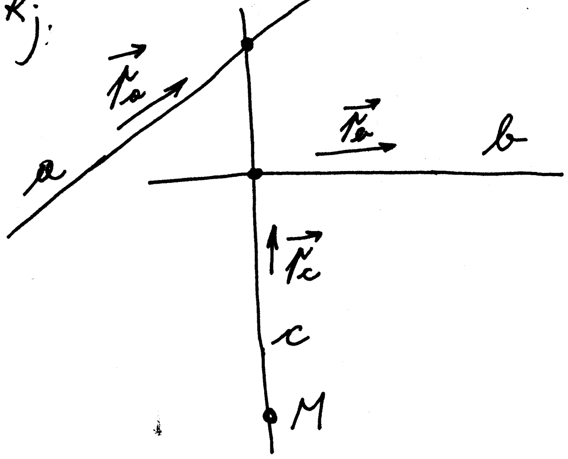
$$H=d=\frac{|(\vec{p}_a\times\vec{p}_a)\cdot\overrightarrow{M_1M_2}|}{|\vec{p}_a\times\vec{p}_a|}=\frac{94}{\sqrt{698}}$$

$$H=\frac{V}{B}$$

(#) Naći jednačinu prave koja prolazi kroz tačku M $M(0,2,-5)$ i siječe prave

$$a: \frac{x-1}{5} = \frac{y+1}{-1} = \frac{z+4}{7} \quad i \quad b: \frac{x+4}{2} = \frac{y-2}{4} = \frac{z+10}{2}.$$

Rj:



$c: ?$

$$c: \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \; (=t$$

$\vec{p}_c = (l, m, n)$

$$c: \begin{cases} x = lt + x_1 \\ y = mt + y_1 \\ z = nt + z_1 \end{cases}$$

$M(0,2,-5)$

$$\text{pa } c: \begin{cases} x = lt \\ y = mt + 2 \\ z = nt - 5 \end{cases}$$ parametarski oblik prave c

$$a: \frac{x-1}{5} = \frac{y+1}{-1} = \frac{z+4}{7} \; (=s)$$

$$b: \frac{x+4}{2} = \frac{y-2}{4} = \frac{z+10}{2} \; (=r)$$

$$\begin{aligned} x-1 &= 5s \\ y+1 &= -s \\ z+4 &= 7s \end{aligned} \Rightarrow a: \begin{cases} x = 5s+1 \\ y = -s-1 \\ z = 7s-4 \end{cases}$$

$$b: \begin{cases} x = 2r-4 \\ y = 4r+2 \\ z = 2r-10 \end{cases}$$ parametarski oblik prave b

Nađimo presječnu tačku pravih a i c.

$$\begin{aligned} 5s+1 &= lt \\ -s-1 &= mt+2 \\ 7s-4 &= nt-5 \end{aligned}$$

$$\begin{aligned} 5s - lt &= -1 \quad (1) \\ -s - mt &= 3 \quad (2) \\ 7s - nt &= -1 \quad (3) \end{aligned}$$

$(1)+5\cdot(2): \; -lt - 5mt = 14$

$(3)+7\cdot(2): \; -nt - 7mt = 20$

$(-l-5m)t = 14$

$(-n-7m)t = 20$

$$t = \frac{14}{-l-5m} = \frac{20}{-n-7m}$$

$-14n - 98m = -20l - 100m \; /:2$

$10l - 7n + m = 0$

$10l + m - 7n = 0$

Pokušajmo naći presječnu tačku pravih b i c:

$$\begin{aligned} 2r-4 &= lt \quad /\cdot 2 \\ 4r+2 &= mt+2 \\ 2r-10 &= nt-5 \quad /\cdot 2 \end{aligned}$$

$$\begin{aligned} 4r &= 2lt + 8 \\ 4r &= mt \\ 4r &= 2nt + 10 \end{aligned}$$

$$\left.\begin{aligned} 2lt+8 &= mt \\ 2nt+10 &= mt \end{aligned}\right\} \rightarrow 2lt+8 = 2nt+10 \quad t = \frac{1}{l-n}$$

$(2l-m)t = -8$

$(2n-m)t = -10$

$$t = \frac{-8}{2l-m}$$

$$t = \frac{-10}{2n-m}$$

Sad možemo formirati jednakost:

$$\frac{-8}{2l-m} = \frac{-10}{2n-m}$$

$-16n + 8m = -20l + 10m$

$20l - 2m - 16n = 0 \; /:2$

$10l - m - 8n = 0$

$$\begin{aligned} 10l + m - 7n &= 0 \quad (I) \\ 10l - m - 8n &= 0 \quad (II) \end{aligned}$$

$(I)+(II): \; 20l - 15n = 0$

$l = \frac{3}{4} n$

$(I)-(II): \; 2m + n = 0$

$m = -\frac{1}{2} n$

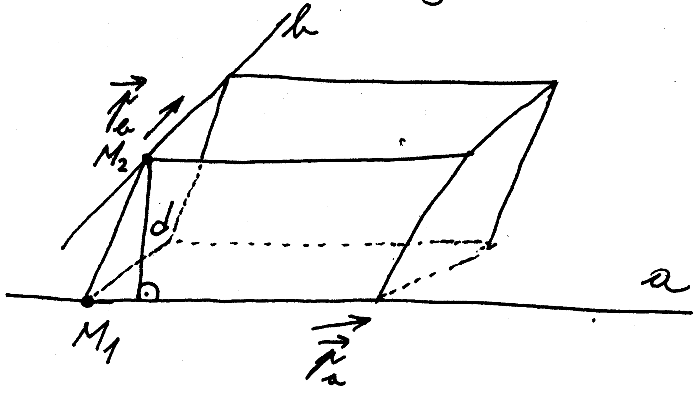
$$\vec{p}_c = \left(\frac{3}{4}n, -\frac{1}{2}n, n\right)_{n \in R}$$

$$\frac{x}{3} = \frac{y-2}{-2} = \frac{z+5}{4}$$

jednačina tražene prave

(#) Izračunati rastojanje između pravih

$$\frac{x-1}{9}=\frac{y}{2}=\frac{z+5}{-4} \quad ; \quad \frac{x}{-6}=\frac{y+4}{-6}=\frac{z-1}{5}$$



Rj. $a: \frac{x-1}{9}=\frac{y}{2}=\frac{z+5}{-4}$

$b: \frac{x}{-6}=\frac{y+4}{-6}=\frac{z-1}{5}$

$\vec{p}_a=(9,2,-4)$ $\quad M_1(1,0,-5)$

$\vec{p}_b=(-6,-6,5)$ $\quad M_2(0,-4,1)$

$\overrightarrow{M_1M_2}(-1,-4,6)$

Zapremina paralopipeda konstruisanog nad vektorima $\vec{p}_a, \vec{p}_b$ i $\overrightarrow{M_1M_2}$ računamo po formuli $|(\vec{p}_a\times\vec{p}_b)\cdot\overrightarrow{M_1M_2}|$.

Zapreminu paralopipeda možemo računati i po formuli $V=B\cdot H$ gdje je B površina paralelograma $|\vec{p}_a\times\vec{p}_b|$

$H=\frac{V}{B}$ tj. $d=\frac{|(\vec{p}_a\times\vec{p}_b)\cdot\overrightarrow{M_1M_2}|}{|\vec{p}_a\times\vec{p}_b|}$ udaljenost između pravih

$$|(\vec{p}_a\times\vec{p}_b)\cdot\overrightarrow{M_1M_2}|=\begin{vmatrix}9&2&-4\\-6&-6&5\\-1&-4&6\end{vmatrix}\ \begin{array}{c}II_k-I_k\cdot4\\ \hline III_k+I_k\cdot6\end{array}\ \begin{vmatrix}9&-34&50\\-6&18&-31\\-1&0&0\end{vmatrix}=(-1)\begin{vmatrix}-34&50\\18&-31\end{vmatrix}=$$

$$=(-1)\,2\begin{vmatrix}-17&25\\18&-31\end{vmatrix}=(-2)(527-450)=(-2)\cdot77=-154$$

$$\vec{p}_a\times\vec{p}_b=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\9&2&-4\\-6&-6&5\end{vmatrix}=-14\vec{i}-21\vec{j}-42\vec{k}=(-14,-21,-42)$$

$$|\vec{p}_a\times\vec{p}_b|=\sqrt{14^2+21^2+42^2}=\sqrt{2^2\cdot7^2+3^2\cdot7^2+6^2\cdot7^2}=7\sqrt{4+9+36}=7\cdot7=49$$

udaljenost je uvijek pozitivna pa $d=\frac{154}{49}\ {=}{:}\ \frac{22}{7}=3\frac{1}{7}$

tražena udaljenost

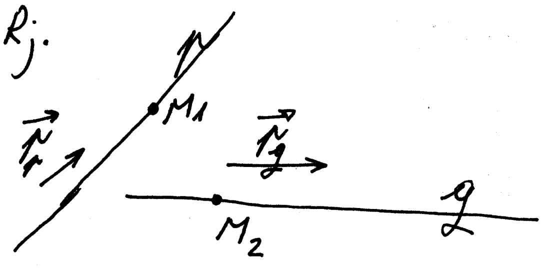
(#) Date su prave $p: \frac{x-4}{1} = \frac{y+3}{2} = \frac{z-12}{-1}$;

$q: \frac{x-3}{-7} = \frac{y-1}{2} = \frac{z-1}{3}$.

a) Utvrditi međusobni položaj pravih p i q.

b) Naći jednačinu zajedničke normale pravih p i q.

Rj.



$\vec{p}_p = (1,2,-1)$

$\vec{p}_q = (-7,2,3)$

$M_1 \in p$

$M_1(4,-3,12)$

$M_2 \in q$

$M_2(3,1,1)$

Ako je $(\vec{p}_p \times \vec{p}_q) \cdot \overrightarrow{M_1M_2} = 0$

$\overrightarrow{M_1M_2} = (-1,4,-11)$

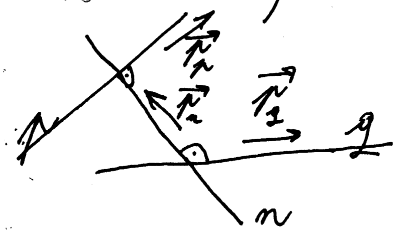
tada su prave p i q komplanarne (nalaze se u istoj ravni)

$$(\vec{p}_p \times \vec{p}_q) \cdot \overrightarrow{M_1M_2} = \begin{vmatrix} 1 & 2 & -1 \\ -7 & 2 & 3 \\ -1 & 4 & -11 \end{vmatrix} \frac{II_v - I_v}{III_v - I_v \cdot 2} \begin{vmatrix} 1 & 2 & -1 \\ -8 & 0 & 4 \\ -3 & 0 & -9 \end{vmatrix} = (-2)\begin{vmatrix} -8 & 4 \\ -3 & -9 \end{vmatrix} =$$

$$= (-2)(-3)(4)\begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix} = 6 \cdot 4 \cdot (-7) \neq 0$$

Prave p i q su dvije mimoilazne prave.

Nađimo zajedničku normalu n pravih p i q



Za vektore pravca važi

$$\left.\begin{matrix} \vec{p}_n \perp \vec{p}_p \\ \vec{p}_n \perp \vec{p}_q \end{matrix}\right\} \Rightarrow \vec{p}_n \parallel \vec{p}_p \times \vec{p}_q$$

$\Downarrow$

$\exists k \in R$, $k \neq 0$ $\vec{p}_n = k(\vec{p}_p \times \vec{p}_q)$

$$\vec{p}_p \times \vec{p}_q = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ -7 & 2 & 3 \end{vmatrix} = 8\vec{i} + 4\vec{j} + 16\vec{k} = 4(2,1,4)$$

$\vec{p}_n = 4k(2,1,4)$, k je neki broj

n: $\frac{x-x_1}{p} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ jednačina prave

$\frac{x-x_1}{4k \cdot 2} = \frac{y-y_1}{4k} = \frac{z-z_1}{4k \cdot 4}$ $\Big/ \cdot 4k$

Trebamo još naći tačku koja pripada pravoj n.

$\frac{x-x_1}{2} = \frac{y-y_1}{1} = \frac{z-z_1}{4}$

Da bi našli tačku $M(x_1, y_1, z_1)$ koja pripada pravoj n prvo ćemo pokušati naći presječne tačke pravih p i n, i pravih g i n i na osnovu toga nešto zaključiti

$$p: \begin{cases} x = t+4 \\ y = 2t-3 \\ z = -t+12 \end{cases} \qquad g: \begin{cases} x = -7s+3 \\ y = 2s+1 \\ z = 3s+1 \end{cases} \qquad n: \begin{cases} x = 2r+x_1 \\ y = r+y_1 \\ z = 4r+z_1 \end{cases}$$

$p \cap n$:

$$t+4 = 2r+x_1 \quad (I)$$
$$2t-3 = r+y_1 \quad (II)$$
$$-t+12 = 4r+z_1 \quad (III)$$

$(I)+(III)$: $16 = 6r+x_1+z_1$

$(II)+2\cdot(III)$: $21 = 9r+y_1+2z_1$

$$r = \frac{16-x_1-z_1}{6} = \frac{21-y_1-2z_1}{9}$$

$$144 \underline{-9x_1} \underline{\underline{-9z_1}} = 126 \underline{\underline{-6y_1}} \underline{\underline{\underline{-12z_1}}}$$

$$-9x_1+6y_1+3z_1+18=0 \quad /:3$$

$$-3x_1+2y_1+z_1+6=0$$

$g \cap n$:

$$-7s+3 = 2r+x_1 \quad (a)$$
$$2s+1 = r+y_1 \quad (b)$$
$$3s+1 = 4r+z_1 \quad (c)$$

$(a)-2(b)$: $-11s+1 = x_1-2y_1$

$(c)-4(b)$: $-5s-3 = z_1-4y_1$

$$s = \frac{1-x_1+2y_1}{11} = \frac{-3-z_1+4y_1}{5}$$

$$5 \underline{-5x_1} \underline{\underline{+10y_1}} = -33 \underline{\underline{\underline{-11z_1}}} \underline{\underline{+44y_1}}$$

$$-5x_1-34y_1+11z_1+38=0$$

$$-3x_1+2y_1+z_1+6=0$$
$$-5x_1-34y_1+11z_1+38=0$$

$$z_1 = 3x_1-2y_1-6$$
$$-5x_1-34y_1+11z_1+38=0$$

$$\underline{-5x_1} \underline{\underline{-34y_1}} \underline{+33x_1} \underline{\underline{-22y_1}} -66+38=0$$

$$28x_1-56y_1-28=0 \quad /:28$$

$$x_1 = 2y_1+1$$

$$z_1 = 3x_1-2y_1-6$$
$$z_1 = 6y_1+3-2y_1-6$$
$$z_1 = 4y_1-3$$

Dobili smo da tačka M ima koordinate $M(2y_1+1, y_1, 4y_1-3)$.

Pokušajmo sad naći presječnu tačku pravih p i n

$$r = \frac{16-x_1-z_1}{6} = \frac{16-2y_1-1-4y_1+3}{6} = \frac{-6y_1+18}{6} = -y_1+3$$

$$t+4 = 2r+x_1 \Rightarrow t = 2(-y_1+3)+2y_1+1-4 = -2y_1+6+2y_1-3 = 3$$

$t=3$ Presječna tačka praveih p i n je $(7, 3, 9)$

Za tačku M mogu uzeti koordinate $(7, 3, 9)$ pa

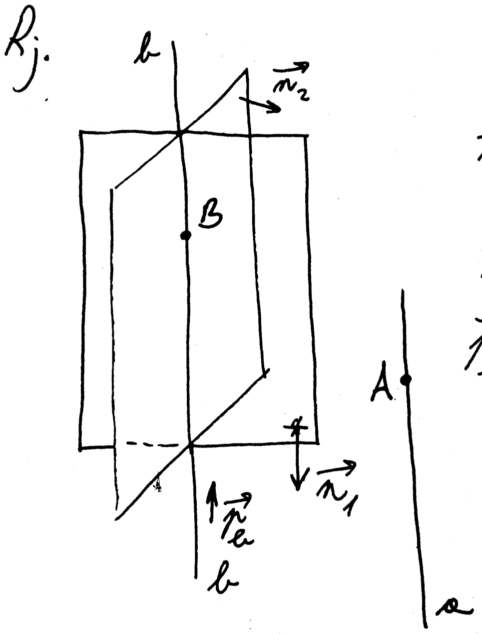
$$\frac{x-7}{2} = \frac{y-3}{1} = \frac{z-9}{4}$$

zajednička normala pravih p i g

(#) Nađi konstante α, β i γ tako da prava

$a: \begin{cases} x = t+2 \\ y = -t-3 \\ z = \gamma t - 1 \end{cases}$ bude paralelna pravoj $b: \begin{cases} \alpha x - 3y - z + 1 = 0 \\ x + \beta y + 2z - 4 = 0 \end{cases}$

Rj.



$\vec{n}_1 = (\alpha, -3, -1)$

$\vec{n}_2 = (1, \beta, 2)$

$\left.\begin{matrix} \vec{p}_b \perp \vec{n}_1 \\ \vec{p}_b \perp \vec{n}_2 \end{matrix}\right\} \Rightarrow \vec{p}_b \parallel \vec{n}_1 \times \vec{n}_2$

$\Downarrow$

$\exists k \in \mathbb{R}\ \ \vec{p}_b = k(\vec{n}_1 \times \vec{n}_2)$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \alpha & -3 & -1 \\ 1 & \beta & 2 \end{vmatrix} =$$

$$= (-6+\beta)\vec{i} - (2\alpha+1)\vec{j} + (\alpha\beta+3)\vec{k}$$

$$= (-6+\beta, -2\alpha-1, \alpha\beta+3)$$

$\Downarrow$

$\vec{p}_b = k(-6+\beta, -2\alpha-1, \alpha\beta+3)$

$k \in \mathbb{R}$

$k \neq 0$

$a: \begin{cases} x = t+2 \\ y = -t-3 \\ z = \gamma t - 1 \end{cases} \Rightarrow \begin{matrix} t = x-2 \\ -t = y+3 \\ \gamma t = z+1 \end{matrix} \Rightarrow \dfrac{x-2}{1} = \dfrac{y+3}{-1} = \dfrac{z+1}{\gamma}$

$\vec{p}_a = (1, -1, \gamma)$

$\vec{p}_a \parallel \vec{p}_b \Rightarrow \exists s \in \mathbb{R}:\ \vec{p}_a = s \cdot \vec{p}_b$

$(1, -1, \gamma) = s \cdot k \cdot (-6+\beta, -2\alpha-1, \alpha\beta+3) \Rightarrow$

$$\Rightarrow \frac{1}{-6+\beta} = \frac{-1}{-2\alpha-1} = \frac{\gamma}{\alpha\beta+3}$$

$6 - \beta = -2\alpha - 1$

$-\beta + 2\alpha = -7$ (a)

$-6\gamma + \beta\gamma = \alpha\beta + 3$

$\alpha\beta - \beta\gamma + 6\gamma = -3$ (b)

$-2\alpha\gamma - \gamma = -\alpha\beta - 3$

$\alpha\beta - 2\alpha\gamma - \gamma = -3$ (c)

(b)-(c): $-\beta\gamma + 2\alpha\gamma + 7\gamma = 0$

$(-\beta + 2\alpha)\gamma + 7\gamma = 0$

(b) i (c) su iste jednakosti

Uvrstimo (a) u (c). Imamo

$\beta = 2\alpha + 7$

$2\alpha^2 + 7\alpha - 2\alpha\gamma - \gamma = -3$

$2\alpha^2 + (7 - 2\gamma)\alpha + 3 - \gamma = 0$

$D = (7 - 2\gamma)^2 - 8(3 - \gamma) =$

$= 49 \underline{- 28\gamma} + 4\gamma^2 - 24 \underline{+ 8\gamma}$

$= 4\gamma^2 - 20\gamma + 25 = (2\gamma - 5)^2$

Kako je $\vec{p}_a \parallel \vec{p}_b \Rightarrow \vec{p}_a \times \vec{p}_b = 0$

$$\vec{p}_a \times \vec{p}_b = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & \gamma \\ -6+\beta & -2\alpha-1 & \alpha\beta+3 \end{vmatrix} = (0, 0, 0)$$

$-2\alpha - 1 - 6 + \beta = 0$

$-2\alpha + \beta = 7$

$-\alpha\beta - 3 = -2\alpha\gamma - \gamma$

$\alpha\beta - 2\alpha\gamma - \gamma = -3$

$\alpha\beta + 3 + 6\gamma - \beta\gamma = 0$

$\alpha\beta - \beta\gamma + 6\gamma = -3$

$$\alpha_{1,2} = \frac{2\gamma - 7 \pm (2\gamma - 5)}{4}$$

$$\alpha_1 = \frac{2\gamma - 7 - 2\gamma + 5}{4} = \frac{-2}{4} = -\frac{1}{2}$$

$$\alpha_2 = \frac{2\gamma - 7 + 2\gamma - 5}{4} = \frac{4\gamma - 12}{4} = \gamma - 3$$

$$2\left(\alpha + \frac{1}{2}\right)(\alpha - \gamma + 3) = 0$$

$$(2\alpha + 1)(\alpha - \gamma + 3) = 0$$

Ako bi α bilo $\alpha = -\frac{1}{2}$ tada bi imali da je $\beta = 6$ pa bi dobili da je $\vec{n}_b = (0,0,0)$ što je nemoguće.

Pa je $\alpha - \gamma + 3 = 0$

$$\alpha = \gamma - 3 \quad \text{tj.} \quad \gamma = \alpha + 3$$

α ću odrediti na sljedeći način. Uzmimo tačku $A \in a$ i tačku $B \in b$. Tada $\vec{AB} \cdot \vec{n_1} = 0$. $(a \parallel b,\ \vec{n}_1 \perp b)$

$A(2,-3,1)$, $A \in a$

$B \in b$, ako uzmem $\gamma = 0$ imamo

$$\alpha x - z + 1 = 0 \quad (\mathrm{I})$$
$$x + 2z - 4 = 0 \quad (\mathrm{II})$$

$(\mathrm{II}) + 2(\mathrm{I})$: $x + 2\alpha x - 2 = 0$

$$(1 + 2\alpha)x = 2$$

$$x = \frac{2}{2\alpha + 1}$$

$$z = \alpha x + 1$$

$$z = \frac{2\alpha}{2\alpha + 1} + \frac{2\alpha + 1}{2\alpha + 1}$$

$$z = \frac{4\alpha + 1}{2\alpha + 1}$$

$$B\left(\frac{2}{2\alpha + 1}, 0, \frac{4\alpha + 1}{2\alpha + 1}\right)$$

$$\frac{2}{2\alpha + 1} - \frac{4\alpha + 2}{2\alpha + 1}$$

$$4\alpha + 1 - 2\alpha - 1$$

$$\vec{AB} = \left(\frac{-4\alpha}{2\alpha + 1}, 3, \frac{2\alpha}{2\alpha + 1}\right)$$

$$\vec{n_1} = (\alpha, -3, -1)$$

$\vec{AB} \cdot \vec{n_1} = 0$ tj. $-\frac{4\alpha}{2\alpha + 1} \cdot \alpha + 3 \cdot (-3) + \frac{2\alpha}{2\alpha + 1} \cdot (-1) = 0 \quad / \cdot (2\alpha + 1)$

$$\alpha \neq -\frac{1}{2}$$

$$-4\alpha^2 - 9(2\alpha + 1) - 2\alpha = 0$$

$$-4\alpha^2 - 20\alpha - 9 = 0$$

$$4\alpha^2 + 20\alpha + 9 = 0$$

$$D = 256$$

$$\alpha_{1,2} = \frac{-20 \pm 16}{8} \Rightarrow \alpha_1 = -\frac{36}{8} \qquad \alpha_2 = -\frac{4}{8}$$

$$\alpha_1 = -\frac{9}{2} \qquad \alpha_2 = -\frac{1}{2}$$

ovo rješenje odbacuj zbog

Tražene konstante α, β i γ su

$$\alpha = -\frac{9}{2}, \quad \beta = -2 \quad \text{i} \quad \gamma = -\frac{3}{2}$$

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

Prava i ravan

Prava $a:\ \frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$, $\vec{p}=(l,m,n)$

Ravan $\alpha:\ Ax+By+Cz+D=0$, $\vec{n}=(A,B\ C)$

1° Ugao između prave a i ravni α $\quad \sin\varphi=\frac{\vec{p}\cdot\vec{n}}{|\vec{p}|\cdot|\vec{n}|}$

uslov paralelnosti $Al+Bm+Cn=0$
$(\vec{p}\perp\vec{n})$

uslov normalnosti $\frac{A}{l}=\frac{B}{m}=\frac{C}{n}$ $\quad(\vec{p}\parallel\vec{n})$

2° Tačka prodora prave i ravni nalazi se tako što se napišu parametarske jednačine prave $x=x_1+lt$, $y=y_1+mt$, $z=z_1+nt$ i zamjene vrijednosti x, y, z u jednačini ravni. Iz tako dobijene jednačine odredi se parametar t a samim tim i koordinate prodora.

3° Uslov da prava a leži u ravni α:

a) $Ax_1+By_1+Cz_1+D=0$ $\quad(M_1(x_1,y_1,z_1)$ tačka na pravoj $a)$,

b) $Al+Bm+Cn=0$

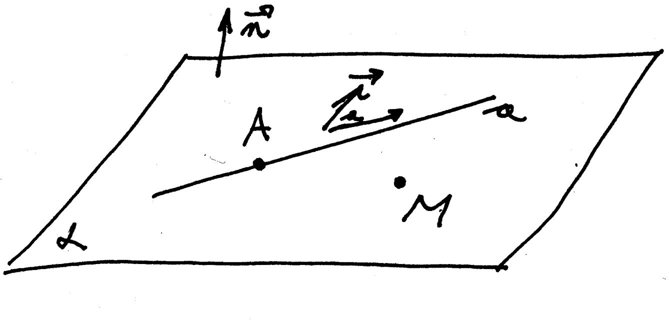
(#) Napisati jednačinu ravni koja sadrži datu tačku $M(4,5,0)$ i datu pravu $\frac{x+3}{5}=\frac{y-4}{-3}=\frac{z-2}{2}$.

Rj:

$a:\ \frac{x+3}{5}=\frac{y-4}{-3}=\frac{z-2}{2}$

$A\in a \quad A(-3,4,2)$

$\vec{p}_a\{5,-3,\ 2\}$



$\alpha=?\quad \alpha:\ A(x-x_1)+B(y-y_1)+C(z-z_1)$

$\vec{n}\{A,B,C\}$

$A(-3,4,2)$
$M(4,5,0)$ $\Rightarrow$ $\overrightarrow{AM}\{7,1,-2\}$ $\left.\begin{matrix}\vec{n}\perp\vec{p}_a\\ \vec{n}\perp\overrightarrow{AM}\end{matrix}\right\}\Rightarrow \vec{n}\parallel \vec{p}_a\times\overrightarrow{AM}$

$$\Downarrow$$

$$\vec{n}=k(\vec{p}_a\times\overrightarrow{AM})$$

$k\in R$

$$\vec{p}_a\times\overrightarrow{AM}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\5&-3&2\\7&1&-2\end{vmatrix}=\vec{i}(6-2)-\vec{j}(-10-14)+\vec{k}(5+21)$$

$$=4\vec{i}+24\vec{j}+26\vec{k}=\{4,24,26\}$$

$$=2\{2,12,13\}$$

Pa mogu uzeti $\vec{n}\{\overset{A}{2},\overset{B}{12},\overset{C}{13}\}$

$\alpha:\ 2(x-4)+12(y-5)+13(z-0)=0$

$2x+12y+13z-8-60=0$ $\quad$ $2x+12y+13z-68=0$ jednačina tražene ravni

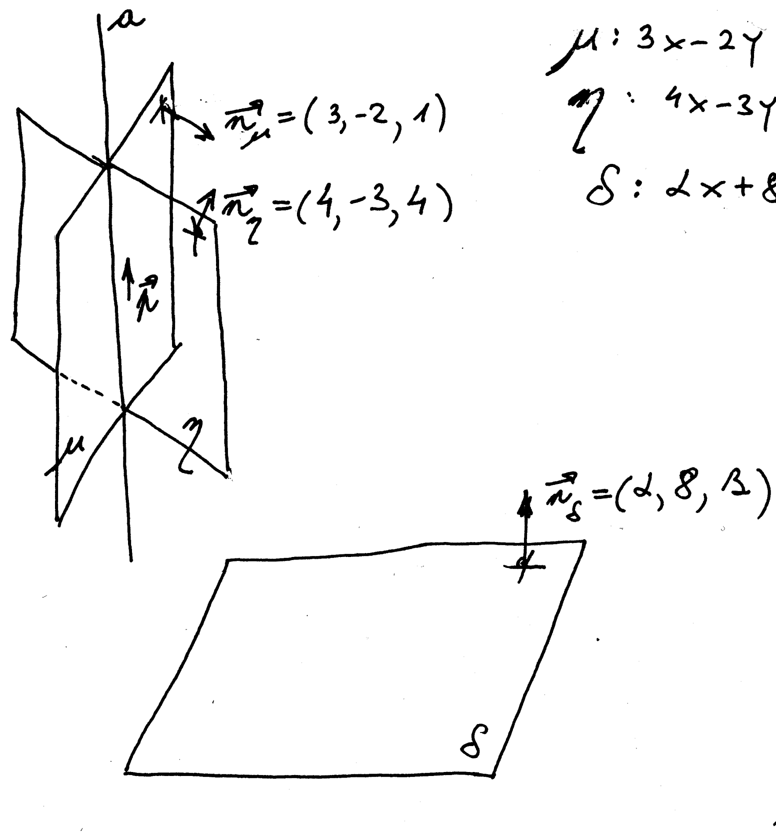
Nađi konstante α i β tako da prava

$$a:\begin{cases}3x-2y+z+3=0\\4x-3y+4z+1=0\end{cases}$$

bude okomita na ravan

$$\alpha x+8y+\beta z+2=0.$$

Rj.



$\mu: 3x-2y+z+3=0$ $\quad$ $\mu\cap\eta=a$

$\eta: 4x-3y+4z+1=0$

$\delta: \alpha x+8y+\beta z+2=0$

$\left.\begin{matrix}\vec{p}\perp\vec{n}_\mu\\ \vec{p}\perp\vec{n}_\eta\end{matrix}\right.\Rightarrow \vec{p}\parallel\vec{n}_\mu\times\vec{n}_\eta$

$$\Downarrow$$

$\exists k\in R$ $\vec{p}=k(\vec{n}_\mu\times\vec{n}_\eta)$

$$\vec{n}_\mu\times\vec{n}_\eta=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\3&-2&1\\4&-3&4\end{vmatrix}=$$

$$=-5\vec{i}-8\vec{j}-\vec{k}=(-5,-8,-1)$$

$$\vec{p}=k(-5,-8,-1)$$

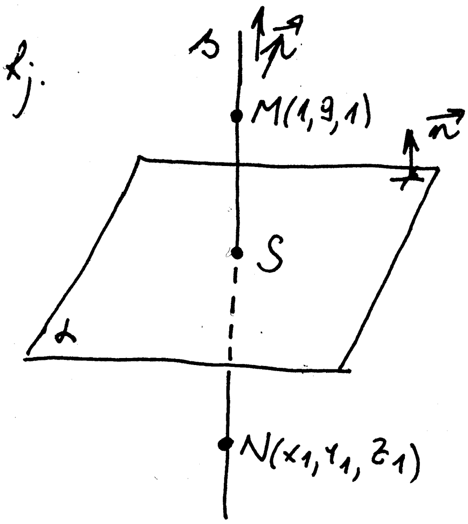
$\vec{p}\parallel\vec{n}_\delta \Rightarrow \exists s\in R: \vec{n}_\delta=s\cdot\vec{p}$

prema tome:
$\vec{n}_\delta=s\cdot k(-5,-8,-1)$

$\vec{n}_\delta=(5,8,1)$ $\Rightarrow$ $\alpha=5,\ \beta=1$ tražene vrijednosti

(#) Odrediti tačku koja je simetrična tački $M(1,9,1)$ u odnosu na ravan $\alpha: 2x+y+3z=0$.

Rj.



$M(1,9,1)$

$\alpha: 2x+y+3z=0$

$M \notin \alpha$

$N=?$ $\qquad |\overrightarrow{MS}| = |\overrightarrow{NS}|$

Da bismo odredili tačku N prvo ćemo postaviti pravu s koja je okomita na α i uz pomoć te prave naći tačku S.

$\vec{n}=(2,1,3)$

$\vec{p} \parallel \vec{n} \Rightarrow$ mogu uzeti $\vec{p}=(2,1,3)$ $\qquad s: \dfrac{x-1}{2} = \dfrac{y-9}{1} = \dfrac{z-1}{3}$ (=t

$$s: \begin{cases} x = 2t+1 \\ y = t+9 \\ z = 3t+1 \end{cases}$$

$x-1=2t$

$y-9=t$

$z-1=3t$

$2x+y+3z=0$

$2(2t+1)+(t+9)+3(3t+1)=0$

$\underline{\underline{4t}}+2+\underline{\underline{t}}+9+\underline{\underline{9t}}+3=0$

$14t=-14$

$t=-1$

Tačka presjeka prave s i ravni α je $S(-1,8,-2)$

$M(1,9,1)$

$S(-1,8,-2)$ $\qquad \overrightarrow{MS}=(-2,-1,-3)$

$N(2t+1, t+9, 3t+1)$

$S(-1, 8, -2)$ $\qquad \overrightarrow{NS}=(-2t-2, -t-1, -3t-3)$

$|\overrightarrow{MS}| = \sqrt{4+1+9} = \sqrt{14}$

$|\overrightarrow{NS}| = \sqrt{(-2t-2)^2+(-t-1)^2+(-3t-3)^2}$

$(-2t-2)^2 = 4t^2+8t+4$

$(-t+1) = t^2+2t+1$

$(-3t-3)^2 = 9t^2+18t+9$

$14t^2+28t+14$

$|\overrightarrow{MS}| = |\overrightarrow{NS}|$

$14t^2+28t+\underline{14} = \underline{14} \quad /:14$

$t^2+2t=0$

$t(t+2)=0$

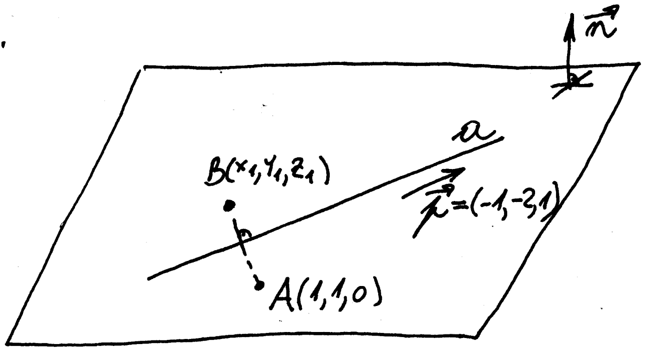
$t=0$ ili $t=-2$

$N(-3,7,-5)$

tražena tačka

(#) Data je prava $a: \frac{x+1}{-1} = \frac{y-2}{-2} = \frac{z}{1}$ i tačka A $A(1,1,0)$. Nađi jednačinu ravni koja sadrži pravu a i tačku A i tačku B simetričnu tački A u odnosu na pravu a.

Rj.



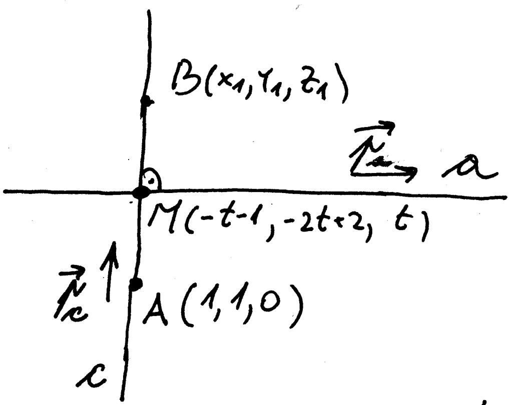
Nađimo prvo tačku $B(x_1,y_1,z_1)$.

$$a: \frac{x+1}{-1} = \frac{y-2}{-2} = \frac{z}{1} \quad (=t)$$

$$a: \begin{cases} x = -t-1 \\ y = -2t+2 \\ z = t \end{cases}$$

Da bi našao tačku B prvo trebamo naći pravu koja prolazi kroz tačke A i B.

$M(-t-1, -2t+2, t)$



$\vec{AM} = (-t-2, -2t+1, t)$

$\vec{p} \perp \vec{AM}$

$\vec{p} \cdot \vec{AM} = 0$ tj. $(-1,-2,1)\cdot(-t-2,-2t+1,t) = 0$

$t+2+4t-2+t=0$

$6t=0 \Rightarrow t=0$

$M(-1, 2, 0)$

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$ jednačina prave kroz dvije tačke

$c: \frac{x-1}{-2} = \frac{y-1}{1} = \frac{z}{0}$, $\vec{p}_c = (-2,1,0)$

Napisaćemo jednačinu ravni koja sadrži pravu a i pravu c (kako ravan sadrži pravu c time će sadržati i tačku B)

$$\left.\begin{matrix} \vec{n} \perp \vec{p} \\ \vec{n} \perp \vec{p}_c \end{matrix}\right\} \Rightarrow \vec{n} \parallel \vec{p} \times \vec{p}_c \Rightarrow \exists k \in R: \vec{n} = k\cdot(\vec{p} \times \vec{p}_c)$$

$$\vec{p} \times \vec{p}_c = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -2 & 1 \\ -2 & 1 & 0 \end{vmatrix} = -\vec{i} - 2\vec{j} - 5\vec{k} = (-1,-2,-5) \Rightarrow \vec{n} = (1,2,5)$$

$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$, jednačina ravni

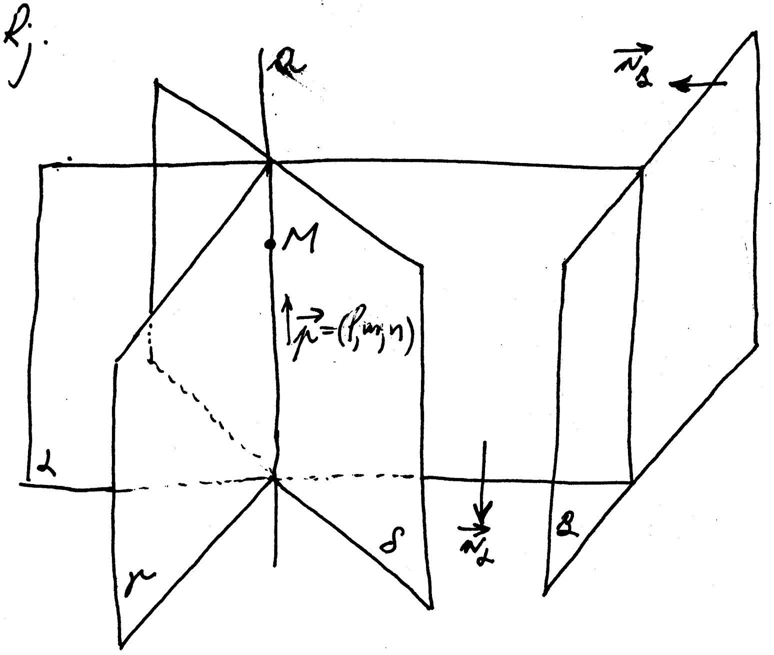
$1(x-1)+2(y-1)+5(z-0)=0$

$x+2y+5z-3=0$ jednačina tražene ravni

#) Napisati jednačinu ravni koja prolazi kroz presjek ravni $\begin{cases} x-y+z+1=0 \\ x+y-z+1=0 \end{cases}$

a normalna je na ravan $2x-y+5z-3=0$.

Rj.



$\alpha: A(x-x_1)+B(y-y_1)+C(z-z_1)=0$

$\beta: 2x-y+5z-3=0$

pramen ravni:

$$A_1x+B_1y+C_1z+D_1 + \\ +\lambda(A_2x+B_2y+C_2z+D_2)=0$$

gdje su

$A_1x+B_1y+C_1z+D_1=0$ i

$A_2x+B_2y+C_2z+D_2=0$

dvije neparalelne ravni koje se sijeku po pravoj.

$$x-y+z+1+\lambda(x+y-z+1)=0$$

$$x+\lambda x-y+\lambda y+z-\lambda z+1+\lambda=0$$

$$x(1+\lambda)+y(-1+\lambda)+z(1-\lambda)+(1+\lambda)=0$$

pramen ravni koje prolaze kroz pravu a

$\vec{n}_\alpha=(1+\lambda,-1+\lambda,1-\lambda)$

$\vec{n}_\beta=(2,-1,5)$

$\vec{n}_\alpha \perp \vec{n}_\beta \Rightarrow \vec{n}_\alpha \cdot \vec{n}_\beta=0$

$(1+\lambda,-1+\lambda,1-\lambda)(2,-1,5)=0$

$2+\underline{2\lambda}+1\underline{-\lambda}+5\underline{-5\lambda}=0$

$-4\lambda+8=0$

$\lambda=2$

$\vec{n}_\alpha=(3,1,-1)$

Treba nam još tačka $M\in a$

$a=\begin{cases} x-y+z+1=0 \\ x+y-z+1=0 \end{cases}$ $(M\in\gamma\wedge\delta)$

$2x+2=0$

$x=-1$

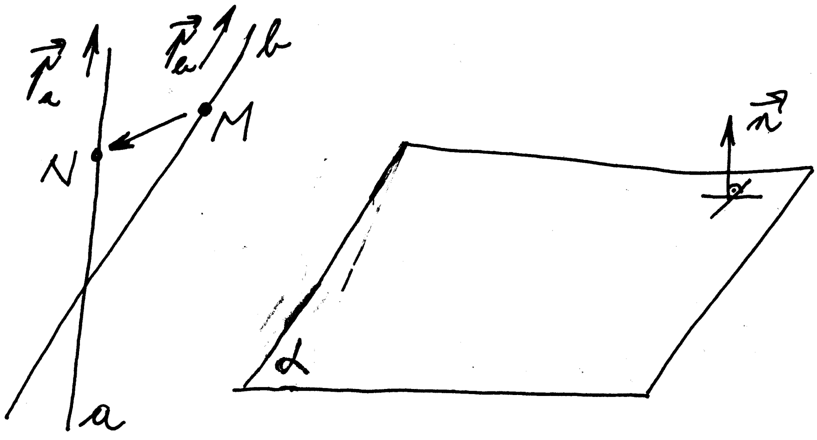
$M(-1,0,0)$

$$3(x+1)+1(y-0)-1(z-0)=0$$

$3x+y-z+3=0$ jednačina tražene ravni

(#) Napisati jednačinu prave koja prolazi kroz tačku $M(3,-2,-4)$, paralelna je ravni $\alpha: 3x-2y-3z-7=0$ i siječe pravu $a: \frac{x-2}{3} = \frac{y+4}{-2} = \frac{z-1}{2}$.

tj.



$$b: \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

$b = ?$ jednačina prave

$M(3,-2,-4)$, $\vec{n}=(3,-2,-3)$

$\vec{p}_a=(3,-2,2)$

$N \in a$, $N(2,-4,1)$

$\overrightarrow{MN}=(-1,-2,-3)$

Vektor $\vec{p}_a$, $\overrightarrow{MN}$ i $\vec{p}_a$ leže u istoj ravni, pa imamo:

$$(\vec{p}_a \times \vec{p}_b)\cdot\overrightarrow{MN}=0 \quad \text{tj.} \quad \begin{vmatrix} 3 & -2 & 2 \\ l & m & n \\ -1 & -2 & -3 \end{vmatrix} = 0$$

$$\vec{p}_b \perp \vec{n} \Rightarrow \vec{p}_a \cdot \vec{n} = 0$$

$$(l,m,n)\cdot(3,-2,-3)=0 \quad \Rightarrow \quad 3l-2m-3n=0$$

$$\begin{vmatrix} 3 & -2 & 2 \\ l & m & n \\ -1 & -2 & -3 \end{vmatrix} = (-1)\begin{vmatrix} 3 & -2 & 2 \\ l & m & n \\ 1 & 2 & 3 \end{vmatrix} \overset{II_k - I_k\cdot 2}{\underset{III_k - I_k\cdot 3}{=}} (-1)\begin{vmatrix} 3 & -8 & -7 \\ l & m-2l & n-3l \\ 1 & 0 & 0 \end{vmatrix} =$$

$$=(-1)\left[-8n+24l-(-7m+14l)\right] = (-1)(-8n\underline{+24l}+7m\underline{-14l})$$

$$=(-1)(10l+7m-8n)=0 \quad \text{tj.} \quad 10l+7m-8n=0$$

$$\begin{aligned} 3l-2m-3n&=0 \quad |\cdot 7 \\ 10l+7m-8n&=0 \quad |\cdot 2 \\ \hline 21l-14m-21n&=0 \\ +\ 20l+14m-16n&=0 \\ \hline 41l-37n&=0 \end{aligned}$$

$$41l=-37n$$

$$l=-\frac{37}{41}n$$

$$2m=3l-3n$$

$$2m=-\frac{111}{41}n-\frac{123}{41}n$$

$$2m=\frac{-234}{41}n \quad |:2$$

$$m=\frac{-117}{41}n$$

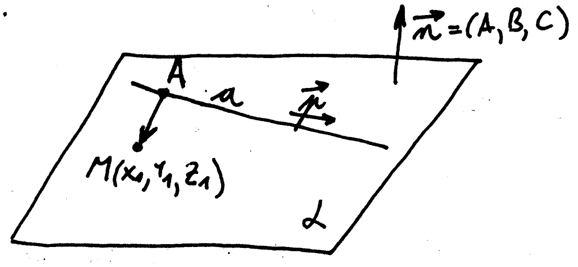
$$\vec{p}_a=\left(-\frac{37}{41}n, \frac{-117}{41}n, n\right)$$

Iz ovoga vidimo da za vektor pravca prave b mogu uzeti:

$\vec{p}_b=(-37,-117,41)$ $\quad b: \frac{x-3}{-37} = \frac{y+2}{-117} = \frac{z+4}{41}$ jednačina tražene prave

(#) Napisati jednačinu ravni koja sadrži tačku $M(1,-1,4)$ i pravu $\frac{x-1}{2}=\frac{y}{1}=\frac{z+1}{3}$.

Rj.



$$\alpha: A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

$$a: \frac{x-1}{2}=\frac{y}{1}=\frac{z+1}{3}$$

$$\vec{p}=(l,m,n)=(2,1,3)$$

$A\in a \quad A(1,0,-1)$

$A(1,0,-1)$
$M(1,-1,4)$ $\quad \overrightarrow{AM}=(0,-1,5)$

$$\left.\begin{array}{l}\overrightarrow{AM}\perp\vec{n}\\ \vec{p}\perp\vec{n}\end{array}\right\} \Rightarrow \vec{n}\parallel \overrightarrow{AM}\times\vec{p}$$

$$\Downarrow$$

$$\vec{n}=k\cdot(\overrightarrow{AM}\times\vec{p}),\ k\in R$$

$$\overrightarrow{AM}\times\vec{p}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\0&-1&5\\2&1&3\end{vmatrix}=\vec{i}(-8)-\vec{j}(-10)+\vec{k}\cdot 2=-8\vec{i}+10\vec{j}+2\vec{k}$$

$$\vec{n}=k(-8,10,2)\Rightarrow \vec{n}=(-4,5,1)$$

$M(1,-1,4)$
$\vec{n}=(-4,5,1)$

$$-4(x-1)+5(y+1)+1(z-4)=0$$

$$-4x+5y+z+4+5-4=0$$

$-4x+5y+z+5=0$ jednačina ravni koja sadrži datu tačku i datu pravu

(#)^{v} Date su ravni $\alpha: x+2y-z-5=0$ i $\beta: x-y+2z-2=0$. Nađi sve tačke na osi Oz koje su podjednako udaljene od ravni α i β.

(#)^{v} Dokazati da su prave $a: \frac{x+1}{3}=\frac{y-2}{2}=\frac{z+4}{1}$; $b: \begin{cases}x-2y+z-3=0\\4x-5y-2z-3=0\end{cases}$ paralelne, pa zatim nađi jednačinu ravni koja ih sadrži.

(#) Kroz središte S duži određene tačkama $A(1,3,0)$ i $B(-3,7,2)$ postaviti pravu p paralelnu pravoj koja je zadana kao presjek ravni $\alpha: 6x-4y+z=16$ i $\beta: y+2z+1=0$.
Prava $g: \begin{cases} x=t+2 \\ y=t+2 \\ z=t+1 \end{cases}$, $t\in\mathbb{R}$ je zadana parametarski. Ispitati odnos između pravih p i g. Ukoliko nisu mimoilazne, napisati jednačinu ravni koja ih sadrži.

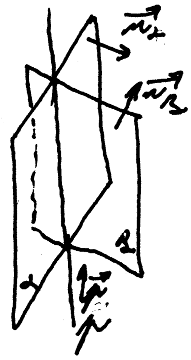
Rj. Nađimo središte S duži AB

$A(1,3,0)$
$B(-3,7,2)$ $\Rightarrow S(-1,5,1)$

$S\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

$\alpha: 6x-4y+z=16$
$\beta: \quad y+2z=-1$

Pronađimo koeficijent pravca prave koja je presjek ove dvije ravni



$\vec{n}_\alpha=(6,-4,1)$

$\vec{n}_\beta=(0,1,2)$

$\left.\begin{array}{l} \vec{p}\perp\vec{n}_\alpha \\ \vec{p}\perp\vec{n}_\beta \end{array}\right\} \Rightarrow \vec{p}\parallel\vec{n}_\alpha\times\vec{n}_\beta$

$\Downarrow$

$\exists k: \vec{p}=k(\vec{n}_\alpha\times\vec{n}_\beta)$

$$\vec{n}_\alpha\times\vec{n}_\beta=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & -4 & 1 \\ 0 & 1 & 2 \end{vmatrix}=$$

$$=-9\vec{i}-12\vec{j}+6\vec{k}$$

$\Downarrow$

$\vec{p}=(-3,-4,2)$

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

jednačina prave kroz jednu tačku

$$\frac{x+1}{-3}=\frac{y-5}{-4}=\frac{z-1}{2}$$ jednačina tražene prave p

$g: \begin{cases} x=t+2 \\ y=t+2 \\ z=t+1 \end{cases}$, $t\in\mathbb{R}$, ...(*) $\quad g: \begin{cases} x-2=t \\ y-2=t \\ z-1=t \end{cases}$, $t\in\mathbb{R}$ $\Rightarrow g: \frac{x-2}{1}=\frac{y-2}{1}=\frac{z-1}{1}$

Koeficijent pravca pave g je $\vec{p}_g=(1,1,1)$.

Prave p i g nisu paralelne (nije $\frac{l_1}{l_2} =: \frac{m_1}{m_2} =: \frac{n_1}{n_2}$)

Pokušajmo naći presječnu tačku pravih p i g.

$p: \begin{cases} x=-3s-1 \\ y=-4s+5 \\ z=2s+1 \end{cases}$, $s\in\mathbb{R}$ ...(**) $\quad$ (*) i (**) $\Rightarrow$

$-3s-1=t+2$ (1)
$-4s+5=t+2$ (2)
$2s+1=t+1$ (3)

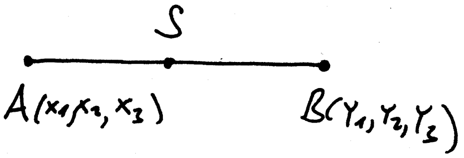
(1)-(2):
$s-6=0$
$s=6 \Rightarrow$
$\Rightarrow t+2=-19$
$t=-\frac{19}{2}$

Kako ovaj t ne zadovoljava (3) sistem nema rješenja.

Prave p i g su mimoilazne.

(#) Kroz središte S duži određene tačkama $A(1,3,0)$ i $B(-3,7,2)$ postaviti pravu p paralelnu pravoj koja je zadana kao presjek ravni $\alpha: 6x-4y+z=16$ i $\beta: y+2z+1=0$.

Rj.



S sredina duži AB

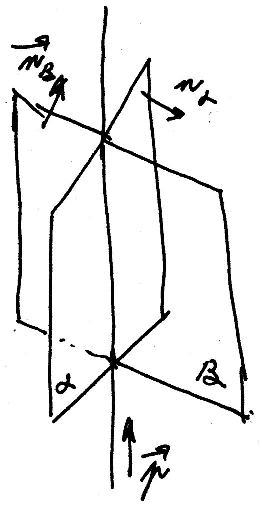
$$S\left(\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2}, \frac{x_3+y_3}{2}\right).$$

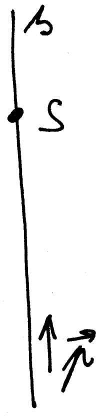
$A(1,3,0)$

$B(-3,7,2)$

$S(-1,5,1)$ sredina duži AB

$$p: \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$ jednačina prave kroz jednu tačku





$\vec{p} = (l, m, n)$

$$\left.\begin{array}{l}\vec{p} \perp \vec{n}_\alpha \\ \vec{p} \perp \vec{n}_\beta\end{array}\right\} \Rightarrow \vec{p} \parallel \vec{n}_\alpha \times \vec{n}_\beta$$

$\Downarrow$

$\exists k: \vec{p} = k(\vec{n}_\alpha \times \vec{n}_\beta)$

$$\vec{n}_\alpha \times \vec{n}_\beta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & -4 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -9\vec{i} - 12\vec{j} + 6\vec{k}$$
$$= (-9, -12, 6)$$

$\vec{n}_\alpha = (6, -4, 1)$

$\vec{n}_\beta = (0, 1, 2)$

Pa za $\vec{p}$ možemo uzeti $\vec{p} = (3, 4, -2)$

$$p: \frac{x+1}{3} = \frac{y-5}{4} = \frac{z-1}{-2}$$ tražena jednačina prave.

Brojni nizovi

Brojni niz je realna f-ja definisana nad skupom prirodnih brojeva.

Npr.

$1, 2, 3, \dots, n, n+1, \dots$ je niz prirodnih brojeva. Opšti član ovog niza je $a_n = n$, $n \in \mathbb{N}$. Niz možemo pisati i u obliku $\{n\}_{n\in\mathbb{N}}$.

$1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \frac{1}{n+1}, \dots$ je niz sa opštim članom $b_n = \frac{1}{n}$, $n \in \mathbb{N}$. Ovaj niz možemo pisati i u obliku $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$

$-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}, \dots$ je niz čiji je opšti član $s_n = \frac{(-1)^n}{n^2}$, $n\in\mathbb{N}$. Skraćeno niz možemo pisati kao $\left\{\frac{(-1)^n}{n^2}\right\}_{n\in\mathbb{N}}$

$\frac{1}{2}, -1, \frac{3}{2}, -2, \frac{5}{2}, -3, \dots$ je niz čiji je opšti član $t_n = \frac{(-1)^{n-1}\cdot n}{2}$. Niz možemo pisati u obliku $\left\{\frac{(-1)^{n-1}}{2}\cdot n\right\}_{n\in\mathbb{N}}$

Aritmetički niz

Aritmetički niz je niz brojeva kod kojih je razlika između dva susjedna člana stalan broj.

$$s+t=n+1$$
$$a_s+a_t=a_1+(s-1)d+a_1+(t-1)d=$$
$$=2a_1+(s+t-2)d=2a_1+(n-1)d$$
$$=a_1+a_n$$

$a_1, a_2, a_3, a_4, \dots, a_n, a_{n+1}, \dots$

$$a_2-a_1=d$$
$$a_3-a_2=d$$
$$a_4-a_3=d$$
$$\vdots$$
$$a_n-a_{n-1}=d$$
$$\vdots$$

$$a_1$$
$$a_2=a_1+d$$
$$a_3=a_2+d=a_1+2d$$
$$a_4=a_3+d=a_1+3d$$
$$\vdots$$
$$a_n=a_{n-1}+d=a_1+(n-1)d$$
$$\vdots$$

$$S_n=a_1+a_2+\dots+a_n$$
$$+\ S_n=a_n+a_{n-1}+\dots+a_1$$
$$2S_n=(a_1+a_n)+(a_2+a_{n-1})+\dots+(a_n+a_1)$$
$$S_n=\frac{n}{2}(a_1+a_n)=\frac{n}{2}(2a_1+(n-1)d)$$

suma prvih n članova

① Izračunati sumu prvih 20 članova niza 2, 5, 8, 11, 14, ...

Rj. Ovo je aritmetički niz, $d=3$

$$a_{20}=a_{19}+3=a_1+19\cdot 3=2+57=59$$

$$S_n=\frac{n}{2}(a_1+a_n)=\frac{20}{2}(2+59)=10\cdot 61=610$$ suma prvih dvadeset članova

Geometriski niz

Geometriski niz je niz brojeva kod kojeg je količnik dva susjedna člana stalan broj.

$b_1, b_2, b_3, b_4, \dots b_{n-1}, b_n, \dots$

$S_n = b_1 + b_2 + b_3 + \dots + b_n$

$b_2 : b_1 = q$

$b_3 : b_2 = q$

$b_4 : b_3 = q$

$\vdots$

$b_n : b_{n-1} = q$

$\vdots$

b_1

$b_2 = b_1 q$

$b_3 = b_2 q = b_1 q^2$

$b_4 = b_3 q = b_1 q^3$

$\vdots$

$b_n = b_{n-1} q = b_1 q^{n-1}$

$\vdots$

$S_n = b_1 + b_1 q + b_1 q^2 + \dots + b_1 q^{n-1}$

$S_n = b_1(1 + q + q^2 + \dots + q^{n-1}) \;/(1-q)$

$(1-q)S_n = b_1(1-q)(1+q+q^2+\dots+q^{n-1})$

$(1-q)S_n = b_1(1-q^n) \quad /:(1-q)$

$S_n = b_1 \dfrac{1-q^n}{1-q}$ suma prvih n članova

② Izračunati sumu prvih 50 članova niza $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$

Rj. Ovo je geometriski niz. $b_1 = \frac{1}{3}$, $q = \frac{1}{3}$, $S_n = b_1 \dfrac{1-q^n}{1-q}$.

$$S_{50} = \frac{1}{3} \cdot \frac{1-(\frac{1}{3})^{50}}{1-\frac{1}{3}} = \frac{1}{3} \cdot \frac{3}{2} \cdot \left(1 - \frac{1}{3^{50}}\right) = \frac{1}{2}\left(1 - \frac{1}{3^{50}}\right) = \frac{1}{2} - \frac{1}{2 \cdot 3^{50}} \approx \frac{1}{2}$$

Monotoni nizovi

Ako je $x_n < x_{n+1}$ tada niz $\{x_n\}_{n\in\mathbb{N}}$ raste

$x_n \leq x_{n+1} \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ ne opada

$x_n > x_{n+1} \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ opada

$x_n \geq x_{n+1} \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ ne raste

ove nizove jednim imenom zovemo monotoni nizovi

$$a_{n+1} - a_n = \dots \begin{cases} <0, & \text{niz opada} \\ >0, & \text{niz raste} \end{cases} \qquad \frac{a_{n+1}}{a_n} = \dots \begin{cases} >1, & \text{rastući niz} \\ <1, & \text{opadajući niz} \end{cases}$$

③ Ispitati monotonost niza $\{a_n\}_{n\in\mathbb{N}}$ gdje je $a_n = \dfrac{n-1}{2n+1}$.

Rj. $$a_{n+1} - a_n = \frac{n+1-1}{2(n+1)+1} - \frac{n-1}{2n+1} = \frac{n}{2n+3} - \frac{n-1}{2n+1} = \frac{2n^2+n-(2n^2-2n+3n-3)}{(2n+3)(2n+1)}$$

$$= \frac{3}{(2n+3)(2n+1)} > 0, \ \forall n$$ $\Rightarrow \{a_n\}$ je rastući niz

Granična vrijednost niza

Broj A nazivamo granična vrijednost niza ili limesom niza realnih brojeva $x_1, x_2, \dots, x_n, \dots$, što simbolički pišemo

$$\lim_{n\to\infty} x_n = A$$

ako za svaki $\varepsilon > 0$ postoji broj N (koji zavisi od ε) tako da

$$|x_n - A| < \varepsilon \quad \text{za svaki } n > N.$$

① Dat je niz $1, \frac{1}{4}, \frac{1}{9}, \dots, \frac{1}{n^2}, \dots$ Izračunati za koju vrijednost n će biti zadovoljena nejednakost $\frac{1}{n^2} < \varepsilon$ ako je $\varepsilon = 0{,}001$.

Rj. $\frac{1}{n^2} < 0{,}001$

$\frac{1}{n^2} < 10^{-3} \quad / \cdot n^2$

$10^{-3} n^2 > 1 \quad / \cdot 10^3$

$n^2 > 10^3$

$n > 10\sqrt{10} \approx 31{,}62$

Za sve $n > 31$ će biti zadovoljena nejednakost $\frac{1}{n^2} < \varepsilon$.

② Pokazati da je $\lim\limits_{n\to\infty} \frac{2n+1}{n+1} = 2$.

Rj. Iz definicije $\forall \varepsilon > 0 \ \exists N$ (koji zavisi od ε) tako da

$$\left| \frac{2n+1}{n+1} - 2 \right| < \varepsilon \quad \text{za svaki } n > N.$$

$$\left| \frac{2n+1}{n+1} - 2 \right| = \left| \frac{2n+1-2n-2}{n+1} \right| = \left| \frac{-1}{n+1} \right| = \frac{1}{n+1} < \varepsilon$$

$(n+1)\varepsilon > 1 \quad / : \varepsilon \ (\varepsilon > 0$

$n+1 > \frac{1}{\varepsilon}$

$n > \frac{1}{\varepsilon} - 1$

Prema tome za svaki pozitivan broj ε postoji takav broj N $(N = \frac{1}{\varepsilon} - 1)$ takav da za $n > N$ vrijedi $\left| \frac{2n+1}{n+1} - 2 \right| < \varepsilon$.

Prema tome $\lim\limits_{n\to\infty} \frac{2n+1}{n+1} = 2$.

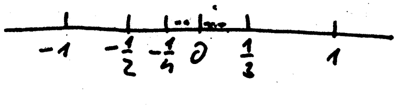
③ Odredite limese nizova

a) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, \frac{(-1)^{n-1}}{n}, \dots$;

b) $\frac{2}{1}, \frac{4}{3}, \frac{6}{5}, \dots, \frac{2n}{2n-1}, \dots$;

c) $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots$;

Rj: a)



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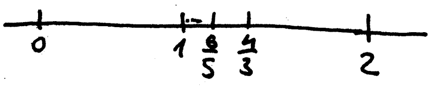
b)



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c) $\sqrt{2} \approx 1{,}41$

$\sqrt{2\sqrt{2}} \approx \sqrt[4]{8} \approx 1{,}68$

$\sqrt{2\sqrt{2\sqrt{2}}} = \sqrt{2\sqrt[4]{8}} =$

$= \sqrt[8]{2^7} \approx 1{,}83$

$\sqrt{2}, \sqrt[4]{2^3}, \sqrt[8]{2^7}, \dots, \sqrt[2^n]{2^{2^n-1}}, \quad \lim\limits_{n\to\infty} 2^{\frac{2^n-1}{2^n}} = 1$

Operacije sa limesima

a) $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$

b) $\lim\limits_{n\to\infty}(a_n \cdot b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

c) $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$

d) $\lim\limits_{n\to\infty} \sqrt[k]{a_n} = \sqrt[k]{\lim\limits_{n\to\infty} a_n}$

e) $\lim\limits_{n\to\infty} b^{a_n} = b^{\lim\limits_{n\to\infty} a_n}, \quad b>0$

f) $\lim\limits_{n\to\infty} \log_b a_n = \log_b \lim\limits_{n\to\infty} a_n, \quad b>1$

① Izračunajte limese

a) $\lim\limits_{n\to\infty} \frac{1}{n}$ Rj. $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

b) $\lim\limits_{n\to\infty} 7$ Rj. $\lim\limits_{n\to\infty} 7 = 7$

c) $\lim\limits_{n\to\infty} n^2$ Rj. $\lim\limits_{n\to\infty} n^2 = \infty$

d) $\lim\limits_{n\to\infty} \frac{n}{n+1}$

Rj. $\lim\limits_{n\to\infty} \frac{n}{n+1} \left(= \frac{\infty}{\infty}\right) \begin{matrix} /:n \\ /:n \end{matrix} = \lim\limits_{n\to\infty} \frac{1}{1+\underset{\to 0}{\boxed{\frac{1}{n}}}} = 1$

e) $\lim\limits_{n\to\infty} \frac{n^2+n-3}{n^3+n^2+1} \begin{matrix} /:n^3 \\ /:n^3 \end{matrix}$ Rj. 0

Neodređeni izrazi sa $\frac{0}{0}, \infty - \infty, 0 \cdot \infty, \frac{\infty}{\infty}, \frac{\infty}{0}$

Određeni izrazi sa $\infty \cdot \infty = \infty, \infty + \infty = \infty, \frac{0}{\infty} = 0$

② Izračunati limese:

a) $\lim\limits_{n\to\infty} \frac{n^3+3n+9}{2n^2+3n-1}$ Rj. $\lim\limits_{n\to\infty} \frac{n^3+3n+9}{2n^2+3n-1} \begin{matrix} /:n^3 \\ /:n^3 \end{matrix} = \lim\limits_{n\to\infty} \frac{1+\overset{\to 0}{\boxed{\frac{3}{n^2}}}+\overset{\to 0}{\boxed{\frac{9}{n^3}}}}{\underset{\to 0}{\boxed{\frac{2}{n}}}+\underset{\to 0}{\boxed{\frac{3}{n^2}}}-\underset{\to 0}{\boxed{\frac{1}{n^3}}}} = \frac{1}{0} = \infty$

b) $\lim\limits_{n\to\infty} \frac{n^2+2n+3}{2n^2+n-4}$ Rj. $\lim\limits_{n\to\infty} \frac{n^2+2n+3}{2n^2+n-4} \begin{matrix} /:n^2 \\ /:n^2 \end{matrix} = \lim\limits_{n\to\infty} \frac{1+\frac{2}{n}+\frac{3}{n^2}}{2+\frac{1}{n}-\frac{4}{n^2}} = \frac{1}{2}$

c) $\lim\limits_{n\to\infty} \frac{3n^3+n-1}{2n^4+1}$ Rj. $\lim\limits_{n\to\infty} \frac{3n^3+n-1}{2n^4+1} \begin{matrix} /:n^4 \\ /:n^4 \end{matrix} = \lim\limits_{n\to\infty} \frac{\frac{3}{n}+\frac{1}{n^3}-\frac{1}{n^4}}{2+\frac{1}{n^4}} = \frac{0}{2} = 0$

d) $\lim\limits_{n\to\infty} \frac{(n+1)(n+2)(n+3)}{n^3}$ Rj. $\lim\limits_{n\to\infty} \frac{(n+1)(n+2)(n+3)}{n^3} \begin{matrix} /:n^3 \\ /:n^3 \end{matrix} = \lim\limits_{n\to\infty} \frac{(1+\frac{1}{n})(1+\frac{2}{n})(1+\frac{3}{n})}{1} = \frac{1}{1} = 1$

e) $\lim\limits_{n\to\infty} \frac{n+(-1)^n}{3n-(-1)^n}$ Rj. $\lim\limits_{n\to\infty} \frac{n+(-1)^n}{3n-(-1)^n} \begin{matrix} /:n \\ /:n \end{matrix} = \lim\limits_{n\to\infty} \frac{1+\overset{0 \leftarrow}{\boxed{\frac{(-1)^n}{n}}}}{3-\underset{\to 0}{\boxed{\left(\frac{(-1)^n}{n}\right)}}} = \frac{1}{3}$

③ Izračunati limese:

a)$^{\vee}$ $\lim\limits_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\dots+\frac{n-1}{n^2}\right)$ b) $\lim\limits_{n\to\infty}\left(\frac{1+3+5+\dots+(2n-1)}{n+1}-\frac{2n+1}{2}\right)$

c)$^{\vee}$ $\lim\limits_{n\to\infty}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots+\frac{1}{2^n}\right)$ d) $\lim\limits_{n\to\infty}\left(1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\dots+\frac{(-1)^{n-1}}{3^{n-1}}\right)$

Rj. a)$^{\vee}$ rj. $\frac{1}{2}$ c)$^{\vee}$ Rj. 1

b) $$\lim_{n\to\infty}\left(\frac{1+3+5+\dots+(2n-1)}{n+1}-\frac{2n+1}{2}\right)=\lim_{n\to\infty}\left(\frac{\frac{n}{2}(1+2n-1)}{n+1}-\frac{2n+1}{2}\right)=\lim_{n\to\infty}\left(\frac{2n^2}{2n+2}-\frac{2n+1}{2}\right)$$
$$=\lim_{n\to\infty}\frac{2n^2-\overset{2n^2+3n+1}{(2n+1)(n+1)}}{2(n+1)}=\lim_{n\to\infty}\frac{\cancel{2n^2}-\cancel{2n^2}-3n-1}{2n+2}\begin{matrix}/:n\\/:n\end{matrix}=\lim_{n\to\infty}\frac{-3-\frac{1}{n}}{2+\frac{2}{n}}=-\frac{3}{2}$$

d) imamo niz $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \dots$ količnik dva susjedna člana je $-\frac{1}{3}$

imamo geometrijski niz, $|q|<1$, $S_n=a_1\frac{1-q^n}{1-q}$

$$\lim_{n\to\infty}\left(1-\frac{1}{3}+\frac{1}{9}-\dots+\frac{(-1)^{n-1}}{3^{n-1}}\right)=\lim_{n\to\infty}\left(1\cdot\frac{1-\boxed{\left(-\frac{1}{3}\right)^n}\to 0}{1-\left(-\frac{1}{3}\right)}\right)=\frac{1}{1+\frac{1}{3}}=\frac{1}{\frac{4}{3}}=\frac{3}{4}$$

④ Izračunati limese:

a) $\lim\limits_{n\to\infty}\frac{2^{n+1}+3^{n+1}}{2^n+3^n}$ b) $\lim\limits_{n\to\infty}(\sqrt{n+1}-\sqrt{n})$ c)$^{\vee}$ $\lim\limits_{n\to\infty}\frac{n\sin n!}{n^2+1}$

d)$^{\vee}$ $\lim\limits_{x\to\infty}\frac{(2x-3)(3x+5)(4x-6)}{3x^3+x-1}$ e)$^{\vee}$ $\lim\limits_{x\to\infty}\frac{1000x}{x^2-1}$ f)$^{\vee}$ $\lim\limits_{x\to\infty}\frac{2x^2-x^3-4}{\sqrt{x^4+1}}$

g)$^{\vee}$ $\lim\limits_{x\to\infty}\frac{2x+3}{x+\sqrt[3]{x}}$ h)$^{\vee}$ $\lim\limits_{x\to\infty}\frac{x^2}{10+x\sqrt{x}}$ i) $\lim\limits_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$

Rj. a) $$\lim_{n\to\infty}\frac{2^{n+1}+3^{n+1}}{2^n+3^n}\begin{matrix}/:3^n\\/:3^n\end{matrix}=\lim_{n\to\infty}\frac{\overset{0\leftarrow}{\boxed{\frac{2^{n+1}}{3^n}}}+3}{\underset{0\leftarrow}{\boxed{\left(\frac{2}{3}\right)^n}}+1}=3$$

b) $$\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\lim_{n\to\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{\infty}=0$$

i) $$\lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}=\lim_{x\to+\infty}\left(\frac{x}{x+\sqrt{x+\sqrt{x}}}\right)^{\frac{1}{2}}=\lim_{x\to+\infty}\left(\frac{1}{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}\right)^{\frac{1}{2}}=1$$

c)$^{\vee}$ rj. 0 d)$^{\vee}$ rj. 8 e)$^{\vee}$ rj. 0 f)$^{\vee}$ rj. 2 g)$^{\vee}$ rj. 2 h) ∞

Granična vrijednost f-je

Kažemo da f-ja $f(x)\to A$ kada $x\to p$ (A i p su brojevi) ili da je $\lim\limits_{x\to a} f(x)=A$ ako za svaki $\varepsilon>0$ postoji takav $\delta>0$ (δ zavisi od ε) da je $|f(x)-A|<\varepsilon$ za $0<|x-p|<\delta$.

① Izračunati limese:

a) $$\lim_{x\to 2}\frac{x^2-4}{x^2-3x+2}\left(=\frac{0}{0}\right)=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(x-1)}=\lim_{x\to 2}\frac{x+2}{x-1}=4$$

b) $$\lim_{x\to -1}\frac{x^3+1}{x^2+1}=\frac{0}{2}=0$$

c) $$\lim_{x\to 5}\frac{x^2-5x+10}{x^2-25}=\frac{25-25+10}{0}=\infty$$

d) $$\lim_{x\to -1}\frac{x^2-1}{x^2+3x+2}\left(=\frac{0}{0}\right)=\lim_{x\to -1}\frac{(x-1)(x+1)}{(x+1)(x+2)}=\lim_{x\to -1}\frac{x-1}{x+2}=\frac{-2}{1}=-2$$

e) $$\lim_{x\to 2}\frac{x^2-2x}{x^2-4x+4}\left(=\frac{0}{0}\right)=\lim_{x\to 2}\frac{x(x-2)}{(x-2)^2}=\lim_{x\to 2}\frac{x}{x-2}=\frac{2}{0}=+\infty$$

f)$^{\vee}$ $$\lim_{x\to 1}\frac{x^3-3x+2}{x^4-4x+3}$$ Rj. $\frac{1}{2}$

g) $$\lim_{x\to a}\frac{x^2-(a+1)x+a}{x^3-a^3}\left(=\frac{0}{0}\right)=\lim_{x\to a}\frac{(x-a)(x-1)}{(x-a)(x^2+ax+a^2)}=\frac{a-1}{a^2+a^2+a^2}=\frac{a-1}{3a^2}$$

$$\begin{array}{l}
(x^2\overbrace{-(a+1)x}^{-ax-x}+a):(x-a)=x-1\\
\underline{-\;\;x^2-ax\qquad\quad}\\
\qquad\qquad -x+a\\
\qquad\qquad \underline{-x+a}\\
\qquad\qquad =\;\;=
\end{array}$$

1
1 1
1 2 1
1 3 3 1

h) $$\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}\left(=\frac{0}{0}\right)=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}=\lim_{h\to 0}(3x^2+3xh+h^2)=3x^2$$

i)$^{\vee}$ $$\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)$$ Rj. -1.

② Izračunati limese

a) $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}\left(=\frac{0}{0}\right)=\left|\begin{matrix}\text{uvodimo smjenu}\\ 1+x=y^6\\ x\to 0\Rightarrow y\to 1\end{matrix}\right|=\lim\limits_{y\to 1}\frac{y^3-1}{y^2-1}=\lim\limits_{y\to 1}\frac{(y-1)(y^2+y+1)}{(y-1)(y+1)}=\frac{3}{2}$

b) $\lim\limits_{x\to 1}\frac{\sqrt{x}-1}{x-1}\left(=\frac{0}{0}\right)=\left|\begin{matrix}x=t^2\\ x\to 1\Rightarrow t\to 1\end{matrix}\right|=\lim\limits_{t\to 1}\frac{t-1}{t^2-1}=\lim\limits_{t\to 1}\frac{t-1}{(t-1)(t+1)}=\frac{1}{2}$

c)$^{\vee}$ $\lim\limits_{x\to 64}\frac{\sqrt{x}-8}{\sqrt[3]{x}-4}$ Rj. 3

d) $\lim\limits_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}\left(=\frac{0}{0}\right)=\left|\begin{matrix}x=t^{12}\\ x\to 1\Rightarrow t\to 1\end{matrix}\right|=\lim\limits_{t\to 1}\frac{\overset{(t^2-1)(t^2+1)}{t^4-1}}{t^3-1}=\lim\limits_{t\to 1}\frac{\cancel{(t-1)}(t+1)(t^2+1)}{\cancel{(t-1)}(t^2+t+1)}=\frac{4}{3}$

e)$^{\vee}$ $\lim\limits_{x\to 1}\frac{\sqrt[3]{x^2}-2\sqrt[3]{x}+1}{(x-1)^2}$ Rj. $\frac{1}{9}$

③ Izračunati limese

a) $\lim\limits_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}\left(=\frac{0}{0}\right)=\lim\limits_{x\to a}\frac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{(x-a)(\sqrt{x}+\sqrt{a})}=\lim\limits_{x\to a}\frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})}=\frac{1}{2\sqrt{a}}$ $(a>0)$

b) $\lim\limits_{x\to 7}\frac{2-\sqrt{x-3}}{x^2-49}\left(=\frac{0}{0}\right)=\lim\limits_{x\to 7}\frac{(2-\sqrt{x-3})(2+\sqrt{x-3})}{\underbrace{(x^2-49)}_{(x-7)(x+7)}(2+\sqrt{x-3})}=\lim\limits_{x\to 7}\frac{\cancel{7-x}}{(-1)\cancel{(7-x)}\underbrace{(x+7)}_{14}\underbrace{(2+\sqrt{x-3})}_{4}}=-\frac{1}{56}$

c)$^{\vee}$ $\lim\limits_{x\to 8}\frac{x-8}{\sqrt[3]{x}-2}$ Rj. 12

d) $\lim\limits_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}\left(=\frac{0}{0}\right)=\lim\limits_{x\to 1}\frac{(\sqrt{x}-1)(\sqrt{x}+1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}{(\sqrt[3]{x}-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)(\sqrt{x}+1)}=\lim\limits_{x\to 1}\frac{\cancel{(x-1)}(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}{\cancel{(x-1)}(\sqrt{x}+1)}=\frac{3}{2}$

e) $\lim\limits_{x\to 4}\frac{3-\sqrt{5+}}{1-\sqrt{5-x}}\left(=\frac{0}{0}\right)=\lim\limits_{x\to 4}\frac{(3-\sqrt{5+x})(3+\sqrt{5+x})(1+\sqrt{5-x})}{(1-\sqrt{5-x})(1+\sqrt{5-x})(3+\sqrt{5+x})}=\lim\limits_{x\to 4}\frac{(4-x)(1+\sqrt{5-x})}{\underbrace{(-4+x)}_{(-1)(4-x)}(3+\sqrt{5+x})}=\frac{2}{-6}=-\frac{1}{3}$

f)$^{\vee}$ $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$ Rj. 1

g) $\lim\limits_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\left(=\frac{0}{0}\right)=\lim\limits_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h\cdot(\sqrt{x+h}+\sqrt{x})}=\lim\limits_{h\to 0}\frac{x+h-x}{h\cdot(\sqrt{x+h}+\sqrt{x})}=\frac{1}{2\sqrt{x}}$

h)$^{\vee}$ $\lim\limits_{h\to 0}\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$ $(x\neq 0)$, Rj. $\frac{1}{3\sqrt[3]{x^2}}$

i)$^{\vee}$ $\lim\limits_{x\to 3}\frac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}$ Rj. $-\frac{1}{3}$

④ Izračunati limese

a) $\lim\limits_{x\to+\infty}(\sqrt{x+a}-\sqrt{x})\ (=\infty-\infty)=\lim\limits_{x\to+\infty}\frac{(\sqrt{x+a}-\sqrt{x})(\sqrt{x+a}+\sqrt{x})}{(\sqrt{x+a}+\sqrt{x})}=\lim\limits_{x\to+\infty}\frac{x+a-x}{(\sqrt{x+a}+\sqrt{x})}=\frac{a}{+\infty}=0$

b) $\lim\limits_{x\to+\infty}\left[\sqrt{x(x+a)}-x\right]\ (=\infty-\infty)=\lim\limits_{x\to+\infty}\frac{[\sqrt{x(x+a)}-x][\sqrt{x(x+a)}+x]}{\sqrt{x(x+a)}+x}=\lim\limits_{x\to+\infty}\frac{x^2+ax-x^2}{\sqrt{x(x+a)}+x}=$

$=\lim\limits_{x\to+\infty}\frac{ax}{\sqrt{x(x+a)}+x}\ \begin{matrix}/:x\\/:x\end{matrix}=\lim\limits_{x\to+\infty}\frac{a}{\sqrt{1\cdot(1+\underbrace{\boxed{\tfrac{a}{x}}}_{\to0})}+1}=\frac{a}{2}$

c)^v $\lim\limits_{x\to+\infty}(\sqrt{x^2-5x+6}-x)$ Rj. $-\frac{5}{2}$

d) $\lim\limits_{x\to+\infty}x(\sqrt{x^2+1}-x)\ (=\infty(\infty-\infty))=\lim\limits_{x\to+\infty}\frac{x(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{(\sqrt{x^2+1}+x)}=\lim\limits_{x\to+\infty}\frac{x(x^2+1-x^2)}{(\sqrt{x^2+1}+x)}$

$=\lim\limits_{x\to\infty}\frac{x}{\sqrt{x^2+1}+x}\ \begin{matrix}/:x\\/:x\end{matrix}=\lim\limits_{x\to\infty}\frac{1}{\sqrt{1+\underbrace{\boxed{\tfrac{1}{x^2}}}_{\to0}}+1}=\frac{1}{2}$

e)^v $\lim\limits_{x\to\infty}(x+\sqrt[3]{1-x^3})$ Rj. 0

Navedimo nekoliko važnih graničnih vrijednosti:

$$\lim_{x\to0}\frac{\sin x}{x}=1 \qquad \lim_{x\to0}(1+x)^{\frac{1}{x}}=e$$

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)=e^k \qquad \lim_{n\to\infty}\sqrt[n]{n}=1$$

$$\lim_{n\to\infty}\sqrt[n]{a}=1 \qquad \lim_{n\to\infty}\frac{a^n}{n}=\infty \qquad \lim_{n\to\infty}\frac{n^k}{a^n}=0$$

⑤ Izračunati limese

a) $\lim\limits_{x\to0}\frac{\sin 5x}{x}=\lim\limits_{x\to0}\left(\frac{\sin 5x}{5x}\cdot5\right)=1\cdot5=5$

b) $\lim\limits_{x\to2}\frac{\sin x}{x}=\frac{1}{2}\sin 2$

c) $\lim\limits_{x\to\infty}\frac{\sin x}{x}=\left|\begin{matrix}\text{kako je}\\-1\le\sin\alpha\le1\\\text{za }\forall\alpha\end{matrix}\right|=0$

d)^v $\lim\limits_{x\to0}\frac{\sin 3x}{x}$ Rj. 3

e) $\lim\limits_{x\to0}\frac{\sin 5x}{\sin 2x}=\lim\limits_{x\to0}\frac{\frac{\sin 5x}{5x}\cdot5}{\frac{\sin 2x}{2x}\cdot2}=\frac{5}{2}$

e) $\lim\limits_{x\to\pi} \frac{\sin mx}{\sin nx} = \left| \begin{matrix} x = \pi + t \\ x\to\pi \Rightarrow t\to 0 \end{matrix} \right| = \lim\limits_{t\to 0} \frac{\sin(m\pi + mt)}{\sin(n\pi + nt)} = \lim\limits_{t\to 0} \frac{\overbrace{\sin m\pi \cos mt}^{=0} + \sin mt \cos m}{\underbrace{\sin n\pi \cos nt}_{=0} + \sin nt \cos n\pi}$

$= \lim\limits_{t\to 0} \frac{(-1)^m \sin mt}{(-1)^n \sin nt} = (-1)^{m-n} \lim\limits_{t\to 0} \frac{\frac{\sin mt}{mt}\cdot mt}{\frac{\sin nt}{nt}\cdot nt} = (-1)^{m-n} \frac{m}{n}$

f) $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \lim\limits_{x\to 0} \frac{2\sin^2\frac{x}{2}}{x^2} = \lim\limits_{x\to 0} \frac{2\left(\sin\frac{x}{2}\right)^2}{4\cdot\left(\frac{x}{2}\right)^2} = \lim\limits_{x\to 0} \frac{1}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}$

$$\left.\begin{matrix} 1 = \sin^2 x + \cos^2 x & & 1 = \sin^2\frac{x}{2} + \cos^2\frac{x}{2} \\ \cos 2x = \cos^2 x - \sin^2 x & & \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} \end{matrix}\right\} \Rightarrow 1 - \cos x = 2\sin^2\frac{x}{2}$$

g) $\lim\limits_{x\to 0} \frac{\operatorname{tg} x}{x} = \lim\limits_{x\to 0} \frac{\frac{\sin x}{\cos x}}{x} = \lim\limits_{x\to 0} \frac{\sin x}{x}\cdot\frac{1}{\cos x} = 1$

h)✓ $\lim\limits_{x\to 1} \frac{\sin\pi x}{\sin 3\pi x}$ Rj. $\frac{1}{3}$ i)✓ $\lim\limits_{n\to\infty}\left(n \sin\frac{\pi}{n}\right)$ Rj π

j) $\lim\limits_{x\to 0} \frac{\sin 5x - \sin 3x}{\sin x} = \lim\limits_{x\to 0} \frac{5\cdot\frac{\sin 5x}{5x} - \frac{\sin 3x}{3x}\cdot 3}{\frac{\sin x}{x}} = 5 - 3 = 2$

k) $\lim\limits_{x\to a} \frac{\sin x - \sin a}{x-a} = \lim\limits_{x\to a} \frac{2\sin\frac{x-a}{2}\cos\frac{x+a}{2}}{x-a} = \lim\limits_{x\to a} \frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}}\cdot\cos\frac{x+a}{2} = \cos a$

$$\left.\begin{matrix} \sin x = \sin\left(\frac{x-a}{2} + \frac{x+a}{2}\right) = \sin\frac{x-a}{2}\cos\frac{x+a}{2} + \sin\frac{x+a}{2}\cos\frac{x-a}{2} \\ -\sin a = \sin(-a) = \sin\left(\frac{x-a}{2} - \frac{x+a}{2}\right) = \sin\frac{x-a}{2}\cos\frac{x+a}{2} - \sin\frac{x+a}{2}\cos\frac{x-a}{2} \end{matrix}\right\} +$$

$\sin x - \sin a = 2\sin\frac{x-a}{2}\cos\frac{x+a}{2}$

⑥ Izračunati limese

a) $\lim\limits_{x\to\infty}\left(\frac{x-1}{x+1}\right)^x = \lim\limits_{x\to\infty}\left(\frac{\frac{x-1}{x}}{\frac{x+1}{x}}\right)^x = \lim\limits_{x\to\infty} \frac{\left(1-\frac{1}{x}\right)^x}{\left(1+\frac{1}{x}\right)^x} = \frac{\lim\limits_{x\to\infty}\left(1+\frac{-1}{x}\right)^x}{e} = \frac{e^{-1}}{e} = e^{-2}$

b) $\lim\limits_{x\to 0}\left(\frac{2+x}{3-x}\right)^x = \left(\frac{2}{3}\right)^0 = 1$

c) $\lim\limits_{x\to\infty}\left(\frac{x+1}{2x+1}\right)^{x^2} = \lim\limits_{x\to\infty}\left(\frac{1+\frac{1}{x}}{2+\frac{1}{x}}\right)^{x^2} = \left(\frac{1}{2}\right)^{+\infty} = 0$

d)✓ $\lim\limits_{x\to 1}\left(\frac{x-1}{x^2-1}\right)^{x+1}$ Rj. $\frac{1}{4}$ e) $\lim\limits_{x\to\infty}\left(\frac{1}{x^2}\right)^{\frac{2x}{x+1}}$ Rj. 0

(#) Izračunati limes $\lim\limits_{n\to\infty}\left(\frac{1+2+3+\ldots+(n-1)}{n+1}-\frac{n}{2}\right)$

Rj.

$1+2+3+\ldots+(n-1) = \frac{n-1}{2}(1+(n-1))$ ← suma aritmetičkog niza

$= \frac{n-1}{2}\cdot n = \frac{n(n-1)}{2}$

$$\lim_{n\to\infty}\left(\frac{1+2+3+\ldots+(n-1)}{n+1}-\frac{n}{2}\right) = \lim_{n\to\infty}\left(\frac{\frac{n(n-1)}{2}}{n+1}-\frac{n}{2}\right) =$$

$$= \lim_{n\to\infty}\left(\frac{n(n-1)}{2(n+1)}-\frac{n}{2}\right) = \lim_{n\to\infty}\frac{n(n-1)-n(n+1)}{2(n+1)} =$$

$$= \lim_{n\to\infty}\frac{n^2-n-n^2-n}{2n+2} = \lim_{n\to\infty}\frac{-2n}{2(n+1)} = \lim_{n\to\infty}\frac{-n \;\; :n}{n+1 \;\; :n}\left(=\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty}\frac{-1}{1+\underset{\swarrow\, 0}{\frac{1}{n}}} = -1$$

(#) Izračunati $\lim\limits_{x\to 1}\frac{\sqrt[3]{x}-1}{1-x}$.

Rj. $a^3-b^3=(a-b)(a^2+ab+b^2)$

$(\sqrt[3]{x}-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)=(\sqrt[3]{x})^3-1^3=x-1$

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{1-x}\left(\frac{0}{0}\right) = -\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}\cdot\frac{(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}{(\sqrt[3]{x^2}+\sqrt[3]{x}+1)} = -\lim_{x\to 1}\frac{\cancel{(x-1)}}{\cancel{(x-1)}(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}$$

$$= -\lim_{x\to 1}\frac{1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1} = \frac{-1}{\sqrt[3]{1}+\sqrt[3]{1}+1} = -\frac{1}{3}$$

Jednostrani limesi

Ako je $x<a$ i $x\to a$, tada po dogovoru pišemo $x\to a-0$; analogno, ako je $x>a$ i $x\to a$, pišemo to ovako $x\to a+0$.

Brojeve $f(a-0)=\lim\limits_{x\to a-0} f(x)$ i $f(a+0)=\lim\limits_{x\to a+0} f(x)$

nazivamo lijevi limes f-je $f(x)$ u tački a i desni limes f-je $f(x)$ u tački a (ako ti brojevi postoje).

Koriste se i sledeće dvije oznake

$$f(a+)=\lim_{x\to a+} f(x) \quad i \quad f(a-)=\lim_{x\to a-} f(x)$$

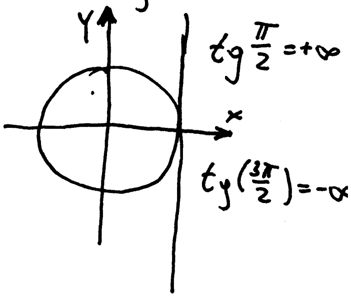
Za postojanje limesa f-je $f(x)$ kada $x\to a$ potrebno je i dovoljno da vrijedi jednakost $f(a-0)=f(a+0)$.

①. Izračunati desni i lijevi limes f-je $f(x)=\operatorname{arc\,tg}\frac{1}{x}$

Rj. $f(+0)=\lim\limits_{x\to +0} \operatorname{arc\,tg}\frac{1}{x}=\frac{\pi}{2}$

$f(-0)=\lim\limits_{x\to -0} \operatorname{arc\,tg}\frac{1}{x}=-\frac{\pi}{2}$

limes f-je $f(x)$ kad $x\to 0$ u ovom slučaju ne postoji



②. Izračunati jednostrane limese

a) $\lim\limits_{x\to -0}\frac{1}{1+e^{\frac{1}{x}}}=\frac{1}{1+e^{-\infty}}=\frac{1}{1+\frac{1}{e^{\infty}}}=1$

b)$^{\vee}$ $\lim\frac{1}{1+e^{\frac{1}{x}}}$ Rj. 0

c) $\lim\limits_{x\to 2+0}\frac{x}{x-2}=\frac{2+0}{2+0-2}=\frac{2+0}{+0}=+\infty$

d)$^{\vee}$ $\lim\limits_{x\to 2-0}\frac{x}{x-2}$ Rj. $-\infty$

e) $\lim\limits_{x\to -0}\frac{|\sin x|}{x}=\lim\limits_{x\to -0}-\frac{\sin x}{x}=-1$

f)$^{\vee}$ $\lim\limits_{x\to +0}\frac{|\sin x|}{x}$ Rj. 1

g) $\lim\limits_{x\to 1-0}\frac{x-1}{|x-1|}=\lim\limits_{x\to 1-0}\frac{(x-1)}{-(x-1)}=\lim\limits_{x\to 1-0}(-1)=-1$

h)$^{\vee}$ $\lim\limits_{x\to 1+0}\frac{x-1}{|x-1|}$ Rj. 1

i) $\lim\limits_{x\to -\infty}\frac{x}{\sqrt{x^2+1}}=\lim\limits_{x\to -\infty}\frac{x}{\sqrt{x^2}}=\lim\limits_{x\to -\infty}\frac{x}{|x|}=\lim\limits_{x\to -\infty}-\frac{x}{x}=-1$

j)$^{\vee}$ $\lim\limits_{x\to +\infty}\frac{x}{\sqrt{x^2+1}}$ Rj. 1

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

Izvod f-je

Definicija Neka je f-ja f definisana na otvorenom intervalu (a,b) i neka je $c\in(a,b)$. Kažemo da f ima izvod (ili derivaciju) u tački c ako postoji limes $\lim\limits_{x\to c}\frac{f(x)-f(c)}{x-c}$. Vrijednost limesa obilježavamo sa $f'(c)$ i zovemo izvod f-je f u tački c.

① Korištenjem navedene definicije naći izvode u tački c sljedećih f-ja:

a) $y=x$ c) $y=\cos x$ e)ˇ $y=x^2$

b) $y=\sqrt[3]{x}$ d)ˇ $y=x^{\alpha}$, $\alpha\in\mathbb{R}$ f)ˇ $y=\sin x$

Rj: a) $f(x)=x$, $f'(c)=\lim\limits_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim\limits_{x\to c}\frac{x-c}{x-c}=\lim\limits_{x\to c}1=1$

$$\Rightarrow (x)'=1$$

b) $f(x)=\sqrt[3]{x}$, $f'(c)=\lim\limits_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim\limits_{x\to c}\frac{\sqrt[3]{x}-\sqrt[3]{c}}{x-c}\cdot\frac{(\sqrt[3]{x^2}+\sqrt[3]{xc}+\sqrt[3]{c^2})}{(\sqrt[3]{x^2}+\sqrt[3]{xc}+\sqrt[3]{c^2})}$

$$=\lim_{x\to c}\frac{\cancel{x-c}}{\cancel{(x-c)}(\sqrt[3]{x^2}+\sqrt[3]{xc}+\sqrt[3]{c^2})}=\frac{1}{3\sqrt[3]{c^2}}\Rightarrow(\sqrt[3]{x})'=\frac{1}{3\sqrt[3]{x^2}}$$

c) $f(x)=\cos x$, $f'(c)=\lim\limits_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim\limits_{x\to c}\frac{\cos x-\cos c}{x-c}\overset{(*)}{=}$

$$\cos x=\cos\frac{x+c+x-c}{2}=\cos\left(\frac{x+c}{2}+\frac{x-c}{2}\right)=\cos\frac{x+c}{2}\cos\frac{x-c}{2}-\sin\frac{x+c}{2}\sin\frac{x-c}{2}$$

$$\cos c=\cos\frac{x+c-x+c}{2}=\cos\left(\frac{x+c}{2}-\frac{x-c}{2}\right)=\cos\frac{x+c}{2}\cos\frac{x-c}{2}+\sin\frac{x+c}{2}\sin\frac{x-c}{2}$$

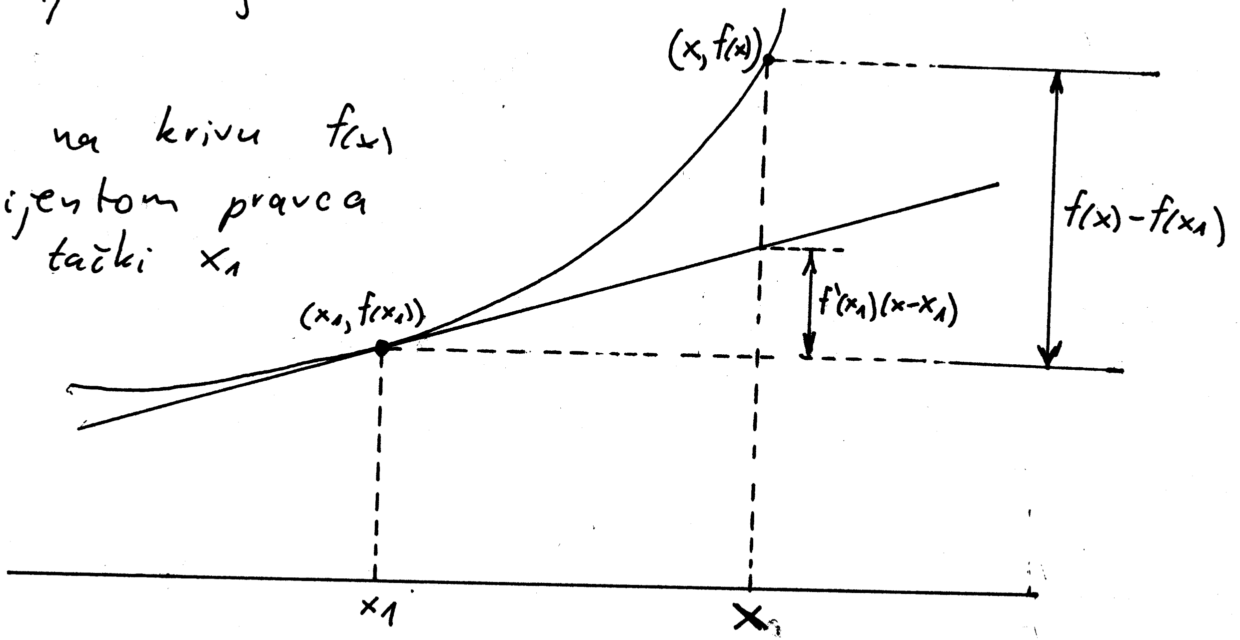
$$\cos x-\cos c=-2\sin\frac{x+c}{2}\sin\frac{x-c}{2}$$

$$\overset{(*)}{=}\lim_{x\to c}\frac{-2\sin\frac{x+c}{2}\sin\frac{x-c}{2}}{x-c}=-\lim_{x\to c}\sin\frac{x+c}{2}\cdot\lim_{x\to c}\cancelto{1}{\frac{\sin\frac{x-c}{2}}{\frac{x-c}{2}}}=-\sin c\Rightarrow(\cos x)'=-\sin x$$

Ako f-ja $f(x)$ ima izvod u tački c tada je $f(x)$ neprekidna u tački c.

Izvodi se upotrebljavaju u mnogim problemima, a najvažnije dvije skupine su:

1. određivanje brzine tačke koja se kreće pravoliniski
2. iznalaženje tangente na krivu

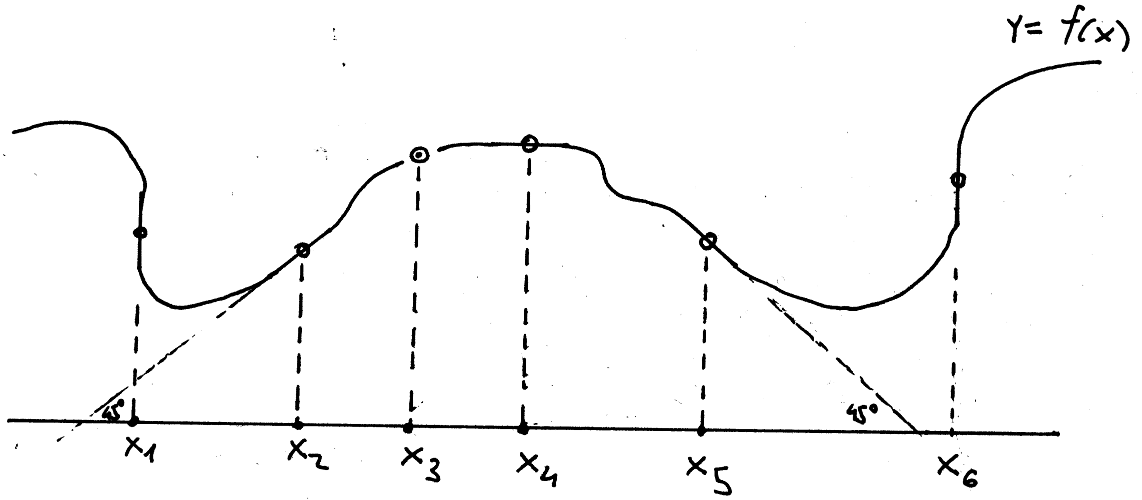


tangenta na krivu $f(x)$ sa koeficijentom pravca $f'(x_1)$ u tački x_1

$$Y - Y_1 = k(x - x_1)$$

$f(x) - f(x_1) = f'(x_1)(x - x_1)$ jednačina tangente na krivu $Y = f(x)$ u nekoj tački $(x_1, f(x_1))$

$k_1 \cdot k_2 = -1$ uslov normalnosti dvije prave



$f'(x_1) = -\infty$

$f'(x_2) = 1$

$f'(x_3)$ ne postoji

$f'(x_4) = 0$

$f'(x_5) = -1$

$f'(x_6) = \infty$

Tablica izvoda

1. $c' = 0$, c - konst.

2. $(x^{\alpha})' = \alpha x^{\alpha-1}, \quad \alpha \in \mathbb{R}$

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}, \quad x > 0$$

3. $(a^x)' = a^x \ln a$

$$(e^x)' = e^x$$

4. $(\log_a x)' = \dfrac{1}{x \ln a}$

$$(\ln x)' = \frac{1}{x}$$

5. $(\sin x)' = \cos x$

6. $(\cos x)' = -\sin x$

7. $(tg\, x)' = \dfrac{1}{\cos^2 x}$

8. $(ctg\, x)' = -\dfrac{1}{\sin^2 x}$

9. $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}, \quad |x| < 1$

10. $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}, \quad |x| < 1$

11. $(arctg\, x)' = \dfrac{1}{1+x^2}$

12. $(arcctg\, x)' = -\dfrac{1}{1+x^2}$

$$\boxed{sh\, x = \frac{e^x - e^{-x}}{2} \qquad ch\, x = \frac{e^x + e^{-x}}{2}}$$

13. $(sh\, x)' = ch\, x$

14. $(ch\, x)' = sh\, x$

15. $(th\, x)' = \dfrac{1}{ch^2 x}$

16. $(cth\, x)' = -\dfrac{1}{sh^2 x}$

Pravila izvoda:

1. $(f \pm g)'(c) = f'(c) \pm g'(c)$

2. $(f \cdot g)'(c) = f'(c)\, g(c) + f(c)\, g'(c)$

3. $(\alpha f)'(c) = \alpha f'(c)$

4. $\left(\dfrac{f}{g}\right)'(c) = \dfrac{f'(c)\, g(c) - f(c)\, g'(c)}{g(c)^2}, \quad g(c) \neq 0$

① Izračunati izvode f-ja:

a) $Y = x^5 - 4x^3 + 2x - 3$

Rj. $Y' = 5x^4 - 12x^2 + 2$

b) $Y = ax^2 + bx + c$

Rj. $Y = 2ax + b$

c) $Y = -\frac{5x^3}{a}$

Rj. $Y' = -\frac{5}{a}(x^3)' = -\frac{15}{a}x^2$

d) $Y = x^2\sqrt[3]{x^2}$

Rj. $Y = x^2 \cdot x^{\frac{2}{3}} = x^{\frac{8}{3}}$

$Y' = \frac{8}{3}x^{\frac{5}{3}} = \frac{8}{3}\sqrt[3]{x^5} = \frac{8}{3}x\sqrt[3]{x^2}$

e) $Y = \frac{a+bx}{c+dx}$

Rj. $Y' = \frac{b(c+dx) - (a+bx)\cdot d}{(c+dx)^2}$

$Y' = \frac{bc + bdx - ad - bdx}{(c+dx)^2}$

$Y' = \frac{bc - ad}{(c+dx)^2}$

g) $Y = \frac{ax^6 + b}{\sqrt{a^2+b^2}}$

Rj. $Y = \frac{a}{\sqrt{a^2+b^2}}x^6 + \frac{b}{\sqrt{a^2+b^2}}$

$Y' = \frac{6a}{\sqrt{a^2+b^2}}x^5$

h) $Y = 3x^{\frac{2}{3}} - 2x^{\frac{5}{2}} + x^{-3}$

Rj. $Y' = 3\cdot\frac{2}{3}x^{-\frac{1}{3}} - 2\cdot\frac{5}{2}x^{\frac{3}{2}} - 3x^{-4}$

$= 2x^{-\frac{1}{3}} - 5x^{\frac{3}{2}} - 3x^{-4}$

i) $Y = \frac{2x+3}{x^2-5x+5}$

Rj. $Y' = \frac{2(x^2-5x+5) - \overbrace{(2x+3)(2x-5)}^{4x^2-10x+6x-15}}{(x^2-5x+5)^2}$

$Y' = \frac{\underline{2x^2}\boxed{-10x}+10\underline{-4x^2}\boxed{+4x}+15}{(x^2-5x+5)^2}$

$Y' = \frac{-2x^2 - 6x + 25}{(x^2-5x+5)^2}$

f) $Y = \frac{2}{2x-1} - \frac{1}{x}$, znamo: $\frac{1}{x} = x^{-1}$

Rj. $Y' = \frac{0(2x-1) - 2(2)}{(2x-1)^2} - (-1)x^{-2}$

$Y' = \frac{-4}{(2x-1)^2} + \frac{1}{x^2}$

$Y' = \frac{1-4x}{x^2(2x-1)^2}$

② Izračunati izvode f-ja:

a)$^{\vee}$ $y = at^m + bt^{m+n}$ Rj. $y' = mat^{m-1} + b(m+n)t^{m+n-1}$

b)$^{\vee}$ $y = \frac{a}{\sqrt[3]{x^2}} - \frac{b}{x\sqrt[3]{x}}$, Rj. $y' = \frac{4b}{3x^2\sqrt[3]{x}} - \frac{2a}{3x\sqrt[3]{x^2}}$

c) $y = \frac{1+\sqrt{z}}{1-\sqrt{z}}$, $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

Rj. $$y' = \frac{\frac{1}{2\sqrt{z}}(1-\sqrt{z}) - (1+\sqrt{z})(-\frac{1}{2\sqrt{z}})}{(1-\sqrt{z})^2} = \frac{\frac{1-\cancel{\sqrt{z}}+1+\cancel{\sqrt{z}}}{2\sqrt{z}}}{(1-\sqrt{z})^2} = \frac{1}{(1-\sqrt{z})^2\sqrt{z}}$$

d) $y = tgx - ctgx$

Rj. $$y' = \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} = \frac{1}{\sin^2 x \cos^2 x} = \frac{4}{(2\sin x\cos x)^2}$$

$$y' = \frac{4}{\sin^2 2x}$$

e)$^{\vee}$ $y = \frac{\pi}{x} + \ln 2$, Rj. $y' = -\frac{\pi}{x^2}$

f) $y = \frac{\sin x + \cos x}{\sin x - \cos x}$

Rj. $$y' = \frac{(\cos x - \sin x)(\sin x - \cos x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$y' = \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2} = \frac{-(\underline{\sin^2 x} - 2\cancel{\sin x}\cos x \boxed{+\cos^2 x} + \underline{\sin^2 x} + 2\cancel{\sin x}\cos x \boxed{+\cos^2 x}}{(\sin x - \cos x)^2}$$

$$y' = \frac{-2}{(\sin x - \cos x)^2}$$

g) $y = 2t\sin t - (t^2-2)\cos t$

Rj. $$y' = 2(1\cdot\sin t + t\cos t) - [2t\cos t + (t^2-2)(-\sin t)] =$$

$$= 2\sin t + 2\cancel{t\cos} t - 2\cancel{t\cos} t + (t^2-2)\sin t = 2\sin t + (t^2-2)\sin t =$$

$$= 2\sin t + t^2\sin t - 2\sin t$$

$$y' = t^2\sin t$$

○ $y = x \arcsin x$

Rj. $y' = \arcsin x + x \cdot \frac{1}{\sqrt{1-x^2}}$

○ $y = (x-1)e^x$

Rj. $y' = e^x + (x-1)e^x$

$y' = e^x(1+x-1) = xe^x$

○ $y = \frac{x^5}{e^x}$

Rj. $y' = \frac{5x^4e^x - x^5e^x}{e^{2x}} = \frac{x^4e^x(5-x)}{(e^x)^2}$

$y' = \frac{x^4(5-x)}{e^x}$

$$\boxed{\log_B A = \frac{\ln A}{\ln B}}$$

○ $y = \frac{x^2}{\ln x}$

Rj. $y' = \frac{2x \cdot \ln x - x^2 \cdot \frac{1}{x}}{\ln^2 x} = \frac{2x\ln x - x}{\ln^2 x}$

$y' = \frac{x(2\ln x - 1)}{\ln^2 x}$

○ $y = \ln x \, \overset{\equiv \log_{10} x}{\boxed{\lg x}} - \ln a \cdot \log_a x$

Rj. $y' = \frac{1}{x}\log_{10} x + \frac{\ln x}{x\ln 10} - \cancel{\ln a}\frac{1}{x\cancel{\ln a}}$

$y' = \frac{1}{x}\frac{\ln x}{\ln 10} + \frac{\ln x}{x \ln 10} - \frac{1}{x}$

$y' = \frac{2\ln x}{x\ln 10} - \frac{1}{x}$

○ ✓ $y = x \operatorname{ctg} x$ Rj. $y' = \operatorname{ctg} x - \frac{x}{\sin^2 x}$

○ ✓ $y = \frac{(1+x^2)\operatorname{arctg} x - x}{2}$ Rj. $y' = x \operatorname{arctg} x$

○ ✓ $y = \frac{1}{x} + 2\ln x - \frac{\ln x}{x}$ Rj. $y' = \frac{2}{x} + \frac{\ln x}{x^2} - \frac{2}{x^2}$

$$\log_B A = \frac{\log_a A}{\log_a B}, \quad \ln x = \log_e x, \quad \log_a B = \frac{1}{\log_B a}$$

Izvodi složenih f-ja

$y=f(\overset{s}{\boxed{g(x)}})$, $y'_x=f'_s\cdot g'_x$ ili $\left.\begin{array}{l} y=\psi(u)\\ u=\varphi(x)\end{array}\right\}$ $y=\psi(\varphi(x))$

① Naći izvode sljedećih f-ja:

$y'_x=y'_u\cdot u'_x$

a) $y=(1+3x-5x^2)^{30}$

Rj. $y=u^{30}$, gdje je $u=1+3x-5x^2$

$y'=30u^{29}\cdot u'$, $u'=3-10x$

$y'=30(1+3x-5x^2)^{30}\cdot(3-10x)$

b) $y=\overset{u}{\boxed{(3+2x^2)}}^4$

Rj. $y'=4(3+2x^2)^3\cdot(3+2x^2)'$

$y'=4(3+2x^2)^3\cdot 4x=16x(3+2x^2)^3$

c) $y=\sqrt[3]{a+bx^3}$

Rj. $y=\sqrt[3]{u}$, $u=a+bx^3$

$y'=\frac{1}{3}u^{-\frac{2}{3}}\cdot u'$, $u'=3bx^2$

$y'=\frac{1}{\cancel{3}u^{\frac{2}{3}}}\cdot\cancel{3}bx^2$

$y'=\frac{bx^2}{\sqrt[3]{(a+bx^3)^2}}$

d)^{v} $f(y)=(2a+3by)^2$

Rj. $f'(y)=12ab+18b^2y$

e) $y=\sqrt{\operatorname{ctg}x}-\sqrt{\operatorname{ctg}\alpha}$

Rj. $y=\sqrt{u}-\sqrt{\operatorname{ctg}\alpha}$, $u=\operatorname{ctg}x$

$y'=\frac{1}{2\sqrt{u}}\cdot u'$, $u'=-\frac{1}{\sin^2x}$

$y'=\frac{-1}{2\sin^2x\sqrt{\operatorname{ctg}x}}$

f) $y=2x+5\cos^3x$

Rj. $y'=2+15\cos^2x\cdot(-\sin x)$

$y'=2-15\cos^2x\sin x$

g)^{v}

$f(x)=-\frac{1}{6(1-3\cos x)^2}$

Rj. $y'=\frac{\sin x}{(1-3\cos x)^3}$

○ Naći izvode sljedećih f-ja:

$$y = x^4(a-2x^3)^2$$

Rj.

$$y' = 4x^3(a-2x^3)^2 + x^4 \cdot 2(a-2x^3) \cdot \overset{(-2)(3)}{(-6)}x^2$$

$$y' = 4x^3(a-2x^3) \cdot \left[\underbrace{a-2x^3 + x \cdot (-1) \cdot 3x^2}_{a-2x^3-3x^3}\right]$$

$$y' = 4x^3(a-2x^3)(a-5x^3)$$

○ $$y = (a+x)\sqrt{a-x}$$

Rj.

$$y' = 1 \cdot \sqrt{a-x} + (a+x)\frac{1}{2\sqrt{a-x}} \cdot (-1)$$

$$y' = \sqrt{a-x} - \frac{a+x}{2\sqrt{a-x}} = \frac{2(a-x)-(a+x)}{2\sqrt{a-x}}$$

$$y' = \frac{a-3x}{2\sqrt{a-x}}$$

○ $$z = \sqrt[3]{y+\sqrt{y}}$$

Rj.

$$(\sqrt[3]{x})' = (x^{\frac{1}{3}})' = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$$

$$z' = \frac{1}{3\sqrt[3]{(y+\sqrt{y})^2}} \cdot (y+\sqrt{y})'$$

$$z' = \frac{1}{3\sqrt[3]{(y+\sqrt{y})^2}} \cdot \left(1 + \frac{1}{2\sqrt{y}}\right)$$

$$z' = \frac{1}{3\sqrt[3]{(y+\sqrt{y})^2}} \cdot \frac{2\sqrt{y}+1}{2\sqrt{y}}$$

$$z' = \frac{2\sqrt{y}+1}{6\sqrt{y}\sqrt[3]{(y+\sqrt{y})^2}}$$

○ $$y = tg^2 5x$$

Rj.

$$y' = 2\,tg5x \cdot (tg5x)'$$

$$y' = 2\,tg5x \cdot \frac{1}{\cos^2 x} \cdot (5x)'$$

$$y' = \frac{10\,tg5x}{\cos^2 x}$$

○ $$y = \sqrt{\cos x} \cdot a^{\sqrt{\cos x}}$$

Rj.

$$y' = \frac{1}{2\sqrt{\cos x}} \cdot (\cos x)' \cdot a^{\sqrt{\cos x}} + \sqrt{\cos x} \cdot a^{\sqrt{\cos x}} \ln a \cdot (\sqrt{\cos x})'$$

$$y' = -\frac{\sin x}{2\sqrt{\cos x}} \cdot a^{\sqrt{\cos x}} + \ln a\sqrt{\cos x} \cdot a^{\sqrt{\cos x}} \cdot \frac{1}{2\sqrt{\cos x}} \cdot (\cos x)'$$

$$y' = -\frac{\sin x}{2\sqrt{\cos x}} a^{\sqrt{\cos x}} - \frac{\ln a \cdot \sin x \cdot \sqrt{\cos x} \cdot a^{\sqrt{\cos x}}}{2\sqrt{\cos x}}$$

$$y' = -\frac{\sin x\, a^{\sqrt{\cos x}}}{2\sqrt{\cos x}}\left[1 + \ln a \cdot \sqrt{\cos x}\right]$$

$$y' = -\frac{\sin x\sqrt{\cos x} \cdot a^{\sqrt{\cos x}}}{2\sqrt{\cos x} \cdot \sqrt{\cos x}}\left[1 + \ln a \cdot \sqrt{\cos x}\right]$$

$$y' = -\frac{1}{2} tg x \cdot y \cdot \left[1 + \ln a\sqrt{\cos x}\right]$$

○ ✓ $$y = 3^{ctg\frac{1}{x}}$$ Rj. $$y' = \frac{3^{ctg\frac{1}{x}} \cdot \ln 3}{(x \sin\frac{1}{x})^2}$$

○ ✓ $$y = \ln(x + \sqrt{a^2+x^2})$$ Rj. $$y' = \frac{1}{\sqrt{a^2+x^2}}$$

\# $y = \ln \frac{(x-2)^5}{(x+1)^3}$

Rj.

$$y = \ln(x-2)^5 - \ln(x+1)^3$$

$$y' = \frac{1}{(x-2)^5} \cdot \left((x-2)^5\right)' - \frac{1}{(x+1)^3} \cdot \left[(x+1)^3\right]'$$

$$y' = \frac{5(x-2)^4}{(x-2)^5} - \frac{3(x+1)^2}{(x+1)^3} \ldots$$

Y mogu napisati i kao

$$y = 5\ln(x-2) - 3\ln(x+1)$$

$$y' = 5 \cdot \frac{1}{x-2} - 3 \cdot \frac{1}{x+1}$$

$$y' = \frac{5(x+1) - 3(x-2)}{(x-2)(x+1)}$$

$$y' = \frac{2x+11}{x^2-x-2}$$

\# $y = \ln \ln(3-2x^3)$

Rj. $y' = \frac{1}{\ln(3-2x^3)} \cdot \left(\ln(3-2x^3)\right)'$

$$y' = \frac{1}{\ln(3-2x^3)} \cdot \frac{1}{3-2x^3} \cdot (\cancel{3}-2x^3)'$$

$$y' = \frac{-6x^2}{(3-2x^3)\ln(3-2x^3)}$$

\# $y = \ln \frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}$

Rj. prvo pojednostavimo izraz

$$\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x} \cdot \frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}+x} =$$

$$= \frac{(\sqrt{x^2+a^2}+x)^2}{x^2+a^2-x^2} = \frac{(\sqrt{x^2+a^2}+x)^2}{a^2}$$

$$y = \ln \frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x} = 2\ln \frac{\sqrt{x^2+a^2}+x}{a^2}$$

$$y' = 2 \cdot \frac{1}{\frac{\sqrt{x^2+a^2}+x}{a^2}} \cdot \left(\frac{\sqrt{x^2+a^2}+x}{a^2}\right)'$$

$$y' = \frac{2\cancel{a^2}}{\sqrt{x^2+a^2}+x} \cdot \frac{1}{\cancel{a^2}} \cdot \left[\frac{1}{\cancel{2}\sqrt{x^2+a^2}} \cdot \overbrace{(x^2+a^2)'}^{\cancel{2}x} + 1\right]$$

$$y' = \frac{2}{\sqrt{x^2+a^2}+x} \cdot \frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}$$

$$y' = \frac{2}{\sqrt{x^2+a^2}}$$

\# $y = \operatorname{arctg} \ln x$

Rj. $y' = \frac{1}{1+\ln^2 x} \cdot (\ln x)'$

$$y' = \frac{1}{x(1+\ln^2 x)}$$

\#$^{\vee}$ $y = \ln \frac{(x-1)^3(x-2)}{x-3}$ Rj. $y' = \frac{3x^2-16x+19}{(x-1)(x-2)(x-3)}$

\#$^{\vee}$ $f(x) = \sqrt{x^2+1} - \ln \frac{1+\sqrt{x^2+1}}{x}$ Rj. $y' = \frac{\sqrt{1+x^2}}{x}$

Izvodi f-ja koje nisu eksplicitno zadane

$y=f(x)$ je eksplicitni oblik f-je. Pored eksplicitnog oblika postoje: $\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$ parametarski oblik f-je

i $F(x,y)=0$ implicitan oblik f-je

① Izračunati $y'=\frac{dy}{dx}$ ako je f-ja y zadana parametarski $\begin{cases} x=a\cos t \\ y=a\sin t \end{cases}$.

Rj. $\frac{dx}{dt}=-a\sin t$ $\quad \frac{dy}{dt}=a\cos t$

$$\frac{dy}{dx}=\frac{a\cos t}{-a\sin t}=-\operatorname{ctg} t$$

tj. $y'=-\operatorname{ctg} t$

② Izračunati $y'=\frac{dy}{dx}$ ako je f-ja y zadana $\begin{cases} x=\sqrt{t} \\ y=\sqrt[3]{t} \end{cases}$.

Rj. $\frac{dx}{dt}=\frac{1}{2\sqrt{t}}$, $\quad \frac{dy}{dt}=\frac{1}{3}t^{-\frac{2}{3}}=\frac{1}{3\sqrt[3]{t^2}}$

$$\frac{dy}{dx}=\frac{\frac{1}{3\sqrt[3]{t^2}}}{\frac{1}{2\sqrt{t}}}=\frac{2\sqrt{t}}{3\sqrt[3]{t^2}}=\frac{2}{3}\sqrt[6]{\frac{t^3}{t^4}}=\frac{2}{3\sqrt[6]{t}}$$

tj. $y'=\frac{2}{3\sqrt[6]{t}}$

③$^{\vee}$ Izračunati $y'=\frac{dy}{dx}$ ako je f-ja y zadana par. $\begin{cases} x=a\cos^3 t \\ y=b\sin^3 t \end{cases}$

Rj. $y'=-\frac{b}{a}\operatorname{tg} t$

④ Izračunati izvod y'_x ako je f-ja zadana implic. $x^3+y^3-3axy=0$.

Rj. $x^3+y^3-3axy=0$

$3x^2+\underline{3y^2\cdot y'}-3ay\underline{-3axy'}=0$

$(3y^2-3ax)y'=3ay-3x^2 \quad /:3$

$$y'=\frac{ay-x^2}{y^2-ax}$$

⑤ Izračunati izvod y'_x ako je f-ja zadana implicitno $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Rj. $\frac{1}{a^2}\cdot 2x+\frac{1}{b^2}\cdot 2y\cdot y'=0$

$\frac{2y}{b^2}y'=-\frac{2x}{a^2} \quad /:2$

$$y'=-\frac{xb^2}{ya^2}$$

⑥$^{\vee}$ Izračunati izvod y'_x ako je f-ja zadana implicitno

$\sqrt{x^2+y^2}=c\cdot\operatorname{arctg}\frac{y}{x}$. $\quad$ Rj. $y'=\frac{cy+x\sqrt{x^2+y^2}}{cx-y\sqrt{x^2+y^2}}$

Logaritamski izvod

Logaritamskim izvodom f-je $y = f(x)$ nazivamo izvodom logaritma te f-je tj. $(\ln y)' = \frac{y'}{y} = \frac{f'(x)}{f(x)}$.

① Naći izvod složene eksplicitno zadane f-je $y = u^v$ ako je $u = \varphi(x)$ i $v = \psi(x)$.

Rj. $y = u^v \quad / \ln$

$\ln y = \ln u^v$

$\ln y = v \ln u \quad /'$

$$\frac{1}{y} \cdot y' = v' \ln u + v \cdot \frac{1}{u} \cdot u' \quad / \cdot y$$

$$y' = y \left(v' \ln u + \frac{v}{u} u' \right)$$

② Izračunati y' ako je $y = (\sin x)^x$.

Rj. $y = (\sin x)^x \quad / \ln$

$\ln y = \ln (\sin x)^x$

$\ln y = x \ln \sin x \quad /'$

$$\frac{1}{y} \cdot y' = \ln \sin x + x \frac{1}{\sin x} \cdot \overset{\cos x}{\overrightarrow{(\sin x)'}}$$

$$y' = y \left(\ln \sin x + x \cdot \frac{\cos x}{\sin x} \right)$$

$$y' = (\sin x)^x (\ln \sin x + x \operatorname{ctg} x)$$

③ Izračunati y' ako je $y = \sqrt[3]{x^2} \cdot \frac{1-x}{1+x^2} \cdot \sin^3 x \cdot \cos^2 x$.

Rj. $\ln y = \ln \sqrt[3]{x^2} + \ln \frac{1-x}{1+x^2} + \ln \sin^3 x + \ln \cos^2 x$

$\ln y = \frac{2}{3} \ln x + \ln \frac{1-x}{1+x^2} + \ln \sin^3 x + \ln \cos^2 x \quad /'$

$$\left(\frac{1-x}{1+x^2} \right)' = \frac{(-1)(1+x^2) - (1-x) \cdot 2x}{(1+x^2)^2} = \frac{-\cancel{1}x^2 - 2x + 2x^2}{(1+x^2)^2} = \frac{x^2 - 2x - 1}{(1+x^2)^2}$$

$$\frac{1}{y} \cdot y' = \frac{2}{3} \cdot \frac{1}{x} + \frac{\cancel{1+x^2}}{1-x} \cdot \frac{x^2-2x-1}{(1+x^2)^{\cancel{2}}} + \frac{3 \cancel{\sin^2 x}}{\sin^{\cancel{3}} x} \cdot \cos x + \frac{2 \cancel{\cos x}}{\cos^{\cancel{2}} x} \cdot (-\sin x)$$

$$y' = y \left(\frac{2}{3x} \cdot \frac{x^2-2x-1}{(1-x)(1+x^2)} + 3 \operatorname{ctg} x - 2 \operatorname{tg} x \right)$$

④ˇ $y = x^x$, Rj. $y' = x^x (1 + \ln x)$

⑤ˇ $y = x^{x^2}$, Rj. $y' = x^{x^2+1} (1 + 2 \ln x)$

⑥ˇ $y = \sqrt[x]{x}$, Rj. $y' = \sqrt[x]{x} \frac{1 - \ln x}{x^2}$

Primjena izvoda u geometriji

Ako je data kriva $Y=f(x)$ i ako je $M(x_1,Y_1)$ data tačka tada je $Y-Y_1=f'(x_1)(x-x_1)$ jednačina tangente u tački M

$x-x_1+f'(x_1)(Y-Y_1)=0$ ili $Y-Y_1=\frac{-1}{f'(x_1)}(x-x_1)$.

je jednačina normale na krivu tački $M(x_1,Y_1)$

Ako su $Y_1=k_1x+n_1$ i $Y_2=k_2x+n_2$ dvije date prave tada je

$tg\varphi=\frac{k_2-k_1}{1+k_1\cdot k_2}$ tangens ugla između dvije prave

Pod uglom između dvije krive $y=f_1(x)$ i $Y=f_2(x)$ u njihovoj presječnoj tački podrazumjevamo uga φ između njihovih zajednički tangenti u presječnoj tački $N(x_1,Y_1)$

$$tg\varphi=\frac{f_2'(x_1)-f_1'(x_1)}{1+f_1'(x_1)\cdot f_2'(x_1)}$$

① Naći jednačinu tangente na krivu $Y=2x^2-4x-6$ u tački $M(\frac{3}{2},\frac{-15}{2})$ i nacrtati sliku.

Rj. $Y=2x^2-4x-6$

nacrtajmo ovu krivu

nule $Y=0$

$2x^2-4x-6=0$

$2(x^2-2x-3)=0$

$2(x+1)(x-3)=0$

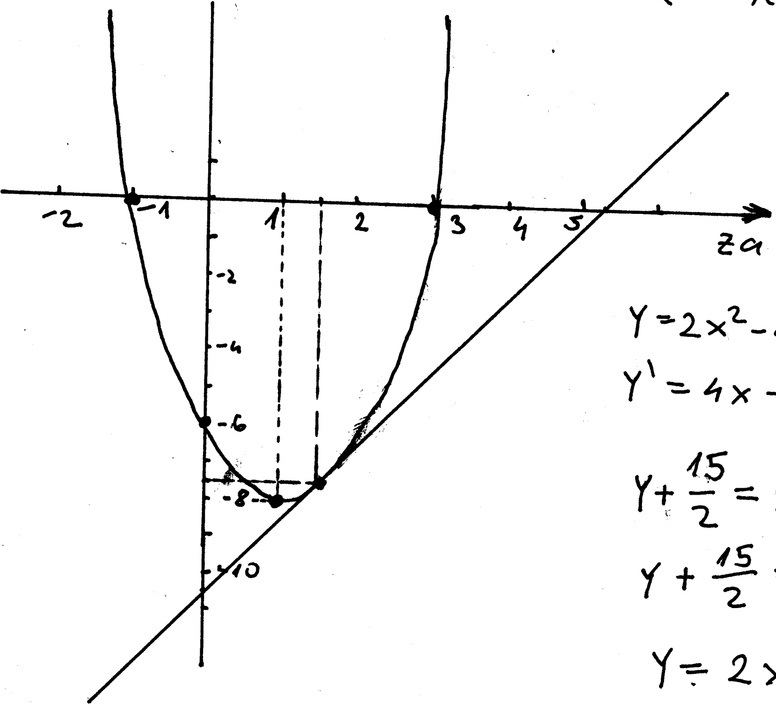
$x_1=3 \Rightarrow Y=0$

$x_2=-1 \Rightarrow Y=0$

tjeme parabole $T\left(-\frac{b}{2a},-\frac{D}{4a}\right)$

$-\frac{b}{2a}=\frac{4}{4}=1$

$-\frac{D}{4a}=-\frac{16+48}{8}=-\frac{64}{8}=-8$



za $x=0 \Rightarrow Y=-6$

$Y=2x^2-4x-6$

$Y'=4x-4$

$M\in f(x)$

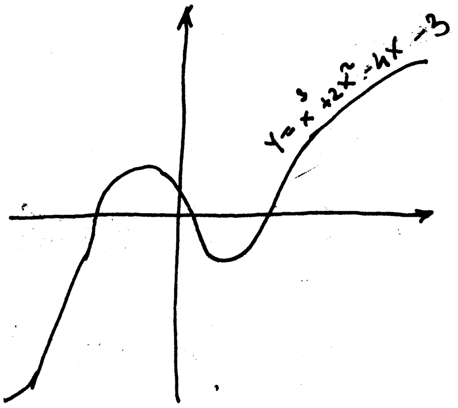
$Y'(\frac{3}{2})=4\cdot\frac{3}{2}-4=6-4=2$

$Y+\frac{15}{2}=2(x-\frac{3}{2})$

$Y+\frac{15}{2}=2x-3$

$Y=2x-\frac{21}{2}$ jednačina tangente

(2.) Napišite jednačinu tangente i normale na krivu $Y=x^3+2x^2-4x-3$ u tački $(-2,5)$.



Rj. $Y' = 3x^2+4x-4$

$Y'(-2)=12-8-4=0$

$Y-Y_0 = f'(x_0)(x-x_0)$

$Y-5=0(x+2)$

$Y-5=0$ jednačina tangente

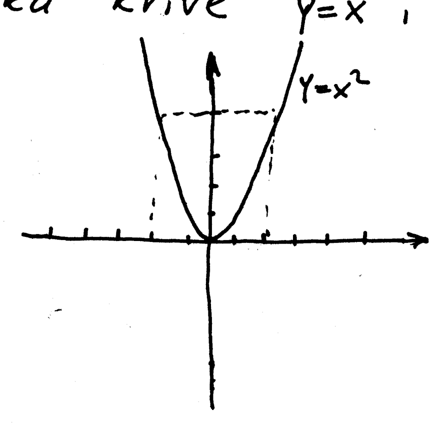
$x-x_0+Y_0'(Y-Y_0)=0$ jedn. norm.

$x+2=0$ jedn. normale

(3.) Nađi jednačinu tangente i normale na krivu $Y=\sqrt[3]{x-1}$ u tački $(1,0)$. Rj. $x-1=0$, $Y=0$

(4.) Odrediti ugao pod kojim se sijeku krive $y=x^2$ i $x=y^2$!

Rj. Prvo nađimo tačke presjeka krivih.



$$\frac{\begin{matrix} Y=x^2 \\ x=Y^2 \end{matrix}}{Y=Y^4}$$

$Y-Y^4=0$

$Y^4-Y=0$

$Y(Y^3-1)=0$

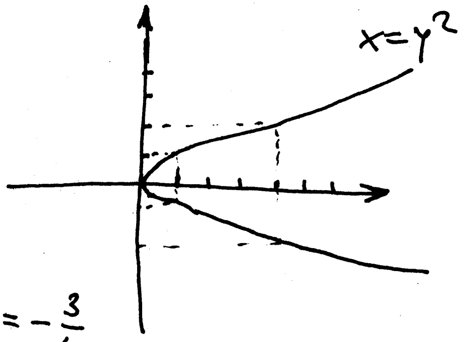
$Y(Y-1)(Y^2+Y+1)=0$

$Y_1=0$ ili $Y_2=1$

$Y_1=0 \Rightarrow x_1=0$

$Y_2=1 \Rightarrow x_2=1$

Postoje dvije tačke presjeka $(0,0)$ i $(1,1)$



f_1: $Y=x^2$

$Y'=2x$

$f_1'(0)=0$

$f_1'(1)=2$

f_2: $x=Y^2$

$1=2YY'$

$Y'=\frac{1}{2Y}$

$f_2'(0)$ nije def.

$f_2'(1)=\frac{1}{2}$

$$tg\varphi = \frac{f_2'(x_0)-f_1'(x_0)}{1-f_1'(x_0)\cdot f_2'(x_0)}$$

$$tg\varphi = \frac{\frac{1}{2}-2}{1+\frac{1}{2}\cdot 2} = \frac{-\frac{3}{2}}{2} = -\frac{3}{4}$$

$\varphi = arc\,tg(-\frac{3}{4})$ ugao pod kojim se sijeku date krive u tački $(1;1)$.

(5.) Nađi ugao pod kojim se sijeku parabole $Y=(x-2)^2$ i $Y=-4+6x-x^2$.

Rj. $\varphi=40°36'$

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

Izvodi višeg reda

$y = f(x)$ – data f-ja

$y' = f'(x)$ prvi izvod

$y'' = (f'(x))' = f''(x)$ drugi izvod

$y''' = \left[f''(x)\right]' = f'''(x)$ treći izvod

$\vdots$

$y^{(n)} = \left[y^{(n-1)}\right]' = f^{(n)}(x)$ n-ti izvod f-je $y = f(x)$

① Naći y''' f-je $y = xe^x$

Rj. $y = xe^x$

$y' = e^x + xe^x = (x+1)e^x$

$y'' = e^x + (x+1)e^x = (x+2)e^x$

$y''' = e^x + (x+2)e^x = (x+3)e^x$

② Naći $y^{(5)}$ f-je $y = 2x^3 + 3x^2 - 4x + 5$

Rj. $y' = 6x^2 + 6x - 4$

$y'' = 12x + 6$

$y''' = 12$

$y^{(4)} = 0$

$y^{(5)} = 0$

③ Naći y'' f-je $y = \ln \dfrac{x^2+3}{x^2+1}$.

Rj.
$$y' = \frac{1}{\frac{x^2+3}{x^2+1}} \cdot \left(\frac{x^2+3}{x^2+1}\right)' = \frac{x^2+1}{x^2+3} \cdot \frac{2x\cdot(x^2+1) - (x^2+3)\cdot 2x}{(x^2+1)^2}$$

$$y' = \frac{2x^3+2x-2x^3-6x}{(x^2+3)(x^2+1)} = \frac{-4x}{(x^2+3)(x^2+1)} = \frac{-4x}{x^4+4x^2+3}$$

$$y'' = \frac{(-4)(x^4+4x^2+3)-(-4x)(4x^3+8x)}{(x^2+3)^2(x^2+1)^2} = \frac{-4x^4-16x^2-12+16x^4+32x^2}{(x^2+3)^2(x^2+1)^2} = \frac{12x^4+16x^2-12}{(x^2+3)^2(x^2+1)^2}$$

$$y'' = \frac{4(3x^4+4x^2-3)}{(x^2+3)^2(x^2+1)^2}$$

④ Nađi y'' f-je $y=(x-1)\,e^{-\frac{1}{x+1}}$

Rj: $y'=\left((x-1)e^{-\frac{1}{x+1}}\right)'=e^{-\frac{1}{x+1}}+(x-1)e^{-\frac{1}{x+1}}\cdot\left(-\frac{1}{x+1}\right)'=$

$$=e^{-\frac{1}{x+1}}+(x-1)\cdot\frac{1}{(x+1)^2}\,e^{-\frac{1}{x+1}}=\left(1+\frac{x-1}{(x+1)^2}\right)e^{-\frac{1}{x+1}}$$

$\left(-\frac{1}{(x+1)}\right)'=\left[-(x+1)^{-1}\right]'=(x+1)^{-2}$ $\qquad y'=\frac{(x+1)^2+x-1}{(x+1)^2}\,e^{-\frac{1}{x+1}}$

$$y'=\frac{x^2+2x\cancel{+1}+x\cancel{-1}}{(x+1)^2}\,e^{-\frac{1}{x+1}}=\frac{x\,(x+3)}{(x+1)^2}\,e^{-\frac{1}{x+1}}=\frac{(x^2+3x)\,e^{-\frac{1}{x+1}}}{x^2+2x+1}$$

$$y''=\left[\frac{x(x+3)\,e^{-\frac{1}{x+1}}}{(x+1)^2}\right]'=\frac{\left[(2x+3)e^{-\frac{1}{x+1}}+(x^2+3x)e^{-\frac{1}{x+1}}\cdot\frac{1}{(x+1)^2}\right]\cdot(x+1)^2-(x^2+3x)e^{-\frac{1}{x+1}}\cdot 2(x+1)}{(x+1)^4}$$

$$y''=\frac{\left[(2x+3)\overbrace{(x+1)^2}^{x^2+2x+1}+x^2+3x-2\overbrace{(x^2+3x)(x+1)}^{x^3+x^2+3x^2+3x}\right]e^{-\frac{1}{x+1}}}{(x+1)^4}$$

$$y''=\frac{\cancel{2x^3}+4x^2\underline{+2x}+3x^2\underline{+6x}+3+x^2\underline{+3x}\cancel{-2x^3}-8x^2\underline{-6x}}{(x+1)^4}\,e^{-\frac{1}{x+1}}$$

$$y''=\frac{5x+3}{(x+1)^4}\,e^{-\frac{1}{x+1}}$$

⑤ Nađi y'' f-ja:

a) $y=\frac{x^3}{x^2-2x-8}$ $\qquad$ Rj: $y''=\frac{24x(x^2+4x+16)}{(x^2-3x-8)^3}$

b) $y=\frac{16}{x^2\cdot(x-4)}$ $\qquad$ Rj: $y''=\frac{64(3x^2-16x+24)}{x^4(x-4)^3}$

c) $y=(2x-1)\,e^{-\frac{x}{x-1}}$ $\qquad$ Rj: $y''=\frac{e^{-\frac{x}{x-1}}}{(x-1)^4}$

L'Hospital-Bernoullijevo pravilo

Ako su obe f-je $f(x)$ i $g(x)$ beskonačno male ili beskonačno velike kad $x\to a$ tj. ako razlomak $\frac{f(x)}{g(x)}$ predstavlja u tački $x=a$ neodređen oblik tipa $\frac{0}{0}$ ili $\frac{\infty}{\infty}$ tada je
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Neodređene limese koji su oblika $0\cdot\infty$, $\infty-\infty$, 1^{∞}, 0^{0}, ∞^{0} skoro uvijek možemo svesti na neki od oblika $\frac{0}{0}$ ili $\frac{\infty}{\infty}$ i onda ih naći pomoću Lopitalovog pravila.

Izračunati:

① $$\lim_{x\to 0}\frac{\ln x}{\operatorname{ctg}x}\left(\frac{-\infty}{\infty}\right)\xlongequal{L_oP_o}\lim_{x\to 0}\frac{(\ln x)'}{(\operatorname{ctg}x)'}=\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{\sin^2x}}=-\lim_{x\to 0}\frac{\sin^2x}{x}=$$
$$=-\lim_{x\to 0}\frac{\sin x}{x}\cdot\sin x=-1\cdot 0=0$$

② $$\lim_{x\to 1}\frac{x^3-2x^2-x+2}{x^3-7x+6}\left(\frac{0}{0}\right)\xlongequal{L_oP_o}\lim_{x\to 1}\frac{3x^2-4x-1}{3x^2-7}=\frac{-2}{-4}=\frac{1}{2}$$

③ $$\lim_{x\to 0}\frac{x\cos x-\sin x}{x^3}\left(\frac{0}{0}\right)\xlongequal{L_oP_o}\lim_{x\to 0}\frac{\overbrace{\cos x+x(-\sin x)-\cos x}^{-x\sin x}}{3x^2}\left(\frac{0}{0}\right)\xlongequal{L_oP}\lim_{x\to 0}\frac{-\sin x+(-x)\cos x}{6x}$$
$$\left(\frac{0}{0}\right)\xlongequal{L_oP_o}\lim_{x\to 0}\frac{-\cos x-\cos x-x(-\sin x)}{6}=\frac{-2}{6}=-\frac{1}{3}$$

④ $$\lim_{x\to 1}\frac{1-x}{1-\sin\frac{\pi x}{2}}\left(\frac{0}{0}\right)\xlongequal{L_oP_o}\lim_{x\to 1}\frac{-1}{-\cos\frac{\pi x}{2}\cdot\frac{\pi}{2}}=\frac{-1}{-0}=+\infty$$

⑤ $$\lim_{x\to 0}\frac{\operatorname{tg}x-\sin x}{x-\sin x}\left(\frac{0}{0}\right)\xlongequal{L_oP_o}\lim_{x\to 0}\frac{\frac{1}{\cos^2x}-\cos x}{1-\cos x}=\lim_{x\to 0}\frac{\frac{1-\cos^3x}{\cos^2x}}{1-\cos x}$$
$$=\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x+\cos^2x)}{\cos^2x\,(1-\cos x)}=3$$

⑥ $$\lim_{x\to 0}\frac{\sin 5x}{x}\left(\frac{0}{0}\right)\xlongequal{L_oP_o}\lim_{x\to 0}\frac{\cos 5x\cdot 5}{1}=5$$

⑦ $\lim\limits_{x\to\infty}\frac{e^x}{x^5}\left(\frac{\infty}{\infty}\right)\overset{L.P.}{=\!=}\lim\limits_{x\to\infty}\frac{e^x}{5x^4}\left(\frac{\infty}{\infty}\right)\overset{L.P.}{=}\lim\limits_{x\to\infty}\frac{e^x}{20x^3}\left(\frac{\infty}{\infty}\right)\overset{L.P.}{=\!=}\dots=\frac{\infty}{120}=\infty$

⑧ $\lim\limits_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}}\left(\frac{\infty}{\infty}\right)\overset{L.P.}{=\!=}\lim\limits_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{3\sqrt[3]{x^2}}}=\lim\limits_{x\to\infty}\frac{3\sqrt[3]{x^2}}{x}=3\lim\limits_{x\to\infty}\frac{x^{\frac{2}{3}}:x}{x\;:x}=3\lim\limits_{x\to\infty}x^{-\frac{1}{3}}=0$

⑨ v $\lim\limits_{x\to 0}\frac{\ln(\sin mx)}{\ln\sin x}$ Rj. 1

⑩ $\lim\limits_{x\to 1}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)\ (\infty-\infty)=\lim\limits_{x\to 1}\frac{\ln x-(x-1)}{(x-1)\ln x}\left(\frac{0}{0}\right)\overset{L.P.}{=\!=}$

$=\lim\limits_{x\to 1}\frac{\frac{1}{x}-1}{\ln x+(x-1)\frac{1}{x}}=\lim\limits_{x\to 1}\frac{\frac{1}{x}-1}{\ln x-\frac{1}{x}+1}\left(\frac{0}{0}\right)\overset{L.P.}{=\!=}\lim\limits_{x\to 1}\frac{-\frac{1}{x^2}}{\frac{1}{x}+\frac{1}{x^2}}=-\frac{1}{2}$

⑪ $\lim\limits_{x\to 0}(1-\cos x)\,\mathrm{ctg}\,x\ (0\cdot\infty)=\lim\limits_{x\to 0}\frac{(1-\cos x)\cos x}{\sin x}\left(\frac{0}{0}\right)=$

$=\lim\limits_{x\to 0}\frac{1-\cos x}{\sin x}\cdot\lim\limits_{x\to 0}\cos x\overset{L.P.}{=\!=}\lim\limits_{x\to 0}\frac{\sin x}{\cos x}\cdot\lim\limits_{x\to 0}\cos x=0\cdot 1=0$

⑫ $\lim\limits_{x\to\infty}\left[x\cdot\left(e^{-\frac{2}{x}}-1\right)\right]\ (\infty\cdot 0)=\lim\limits_{x\to\infty}\frac{e^{-\frac{2}{x}}-1}{\frac{1}{x}}\left(\frac{0}{0}\right)\overset{L.P.}{=\!=}\lim\limits_{x\to\infty}\frac{e^{-\frac{2}{x}}\cdot(-2)\cdot(-1)\cdot x^{-2}}{(-1)\cdot x^{-2}}$

$=e^0\cdot(-2)=-2$

⑬ v $\lim\limits_{x\to\infty}x\cdot\sin\frac{a}{x}$ Rj. a

⑭ $\lim\limits_{x\to 1}x^{\frac{1}{1-x}}\ (1^\infty)=\lim\limits_{x\to 1}e^{\ln x^{\frac{1}{1-x}}}=\lim\limits_{x\to 1}e^{\frac{1}{1-x}\cdot\ln x}=e^{\lim\limits_{x\to 1}\frac{\ln x}{1-x}}\left(\frac{0}{0}\right)=$

$\overset{L.P.}{=\!=}e^{\lim\limits_{x\to 1}\frac{\frac{1}{x}}{-1}}=e^{-1}=\frac{1}{e}$

⑮ $\lim\limits_{x\to 0}(\mathrm{ctg}\,x)^{\frac{1}{\ln x}}\ (\infty^\infty)=\lim\limits_{x\to 0}e^{\ln(\mathrm{ctg}\,x)^{\frac{1}{\ln x}}}=e^{\lim\limits_{x\to 0}\frac{\ln(\mathrm{ctg}\,x)}{\ln x}}\left(\frac{\infty}{\infty}\right)$

$\overset{L.P.}{=\!=}e^{\lim\limits_{x\to 0}\frac{\frac{1}{\mathrm{ctg}\,x}\cdot-\frac{1}{\sin^2 x}}{\frac{1}{x}}}=e^{\lim\limits_{x\to 0}\frac{-\frac{\sin x}{\cos x}\cdot\frac{1}{\sin^2 x}}{\frac{1}{x}}}=e^{\lim\limits_{x\to 0}\frac{-x}{\sin x\cos x}}\left(\frac{0}{0}\right)\overset{L.P.}{=\!=}e^{\lim\limits_{x\to 0}\frac{-1}{\cos^2 x-\sin^2 x}}$

$=e^{-1}=\frac{1}{e}$

⑯ v $\lim\limits_{x\to 0}x^{\sin x}$ Rj. 1

⑰ v $\lim\limits_{x\to\infty}\left[(x-1)e^{\frac{-1}{x+1}}-x\right]$ Rj. **−2**

(#) Ako je $h(x)=\frac{1}{\sin x}-\frac{1}{x}$ izračunati $\lim\limits_{x\to 0} h'(x)$.

Rj. $h(x)=\frac{1}{\sin x}-\frac{1}{x}=\frac{x-\sin x}{x\sin x}$

$$h'(x)=\left(\frac{1}{\sin x}\right)'-\left(\frac{1}{x}\right)'=(\sin^{-1}x)'-(x^{-1})'=(-1)\sin^{-2}x\cdot\cos x-(-1)\,x^{-2}$$

$$h'(x)=\frac{-\cos x}{\sin^2 x}+\frac{1}{x^2}=\frac{1}{x^2}-\frac{\cos x}{\sin^2 x}=\frac{\sin^2 x-x^2\cos x}{x^2\sin^2 x}$$

$$\lim_{x\to 0} h'(x)=\lim_{x\to 0}\frac{\sin^2x-x^2\cos x}{x^2\sin^2 x}\left(=\frac{0}{0}\right)\overset{L_oP_o}{=}$$

$$\overset{L_oP_o}{=}\lim_{x\to 0}\frac{\overbrace{2\sin x\cos x}^{\sin 2x}-(2x\cos x+x^2(-\sin x))}{2x\sin^2x+x^2\underbrace{2\sin x\cos x}_{\sin 2x}}=\lim_{x\to 0}\frac{\sin 2x-2x\cos x+x^2\sin x}{2x\sin^2x+x^2\sin 2x}$$

$$\left(=\frac{0}{0}\right)\overset{L_oP_o}{=}\lim_{x\to 0}\frac{\cos 2x\cdot 2-2(\cos x+x(-\sin x))+(2x\sin x+x^2\cos x)}{2(\sin^2x+x\underbrace{2\sin x\cos x}_{\sin 2x})+2x\sin 2x+x^2\cos 2x\cdot 2}=$$

$$=\lim_{x\to 0}\frac{2\cos 2x-2\cos x+2x\sin x+2x\sin x+x^2\cos x}{2\sin^2x+\underline{\underline{2x\sin 2x}}+\underline{\underline{2x\sin 2x}}+2x^2\cos 2x}=$$

$$=\lim_{x\to 0}\frac{2\cos 2x-2\cos x+x^2\cos x+4x\sin x}{2\sin^2x+2x^2\cos 2x+4x\sin 2x}$$

$$=\lim_{x\to 0}\frac{2(-\sin 2x)\cdot 2-2(-\sin x)+(2x\cos x+x^2(-\sin x))+4\sin x+4x\cos x}{2\cdot\underbrace{2\sin x\cos x}_{\sin 2x}+2(2x\cos 2x+x^2(-\sin 2x)\cdot 2)+4\sin 2x+4x\cos 2x\cdot 2}$$

$$=\lim_{x\to 0}\frac{-4\sin 2x+6\sin x+6x\cos x-x^2\sin x}{6\sin 2x+12x\cos 2x-4x^2\sin 2x}\left(=\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{-4\cos 2x\cdot 2+6\cos x+6(\cos x+x(-\sin x))\cdot(2x\sin x+x^2\cos x)}{6\cos 2x\cdot 2+12(\cos 2x+x(-\sin 2x)\cdot 2)-4(2x\sin 2x+x^2\cos 2x\cdot 2)}=$$

$$=\frac{-8+6+6}{12+12}=\frac{4}{24}=\frac{1}{6}$$

Prema tome $\lim\limits_{x\to 0} h'(x)=\frac{1}{6}$

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

Ispitivanje f-je

Ispitati f-ju znači odrediti

a) oblast definisanosti
b) parnost (neparnost) i periodičnost
c) nule, presjek grafa sa Y-osom, znak f-je
d) ponašanje na krajevima intervala definisanosti i asimptote
e) rast i opadanje f-je (intervale u kojima f-ja raste ili opada)
f) ekstreme f-je (minimum i maksimum ako ih ima)
g) prevojne tačke i intervale konveksnosti i konkavnosti
h) na osnovu svega ovoga nacrtati graf

Definiciono područje obilježavat ćemo sa $\mathcal{D}$ i to je skup svih onih vrijednosti u kojima je f-ja definisana (ima konačnu ili beskonačnu vrijednost).

① Odrediti definiciono područje sljedećih f-ja:

a) $Y=\frac{1}{x}$, Rj. $\mathcal{D}: \mathbb{R}\setminus\{0\}$ ili $\mathcal{D}: x\in(-\infty,0)\cup(0,+\infty)$

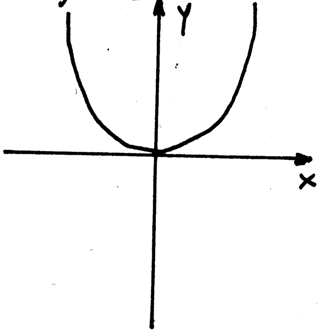
b) $Y=\sqrt{x}$, Rj. $\mathcal{D}: x\in\mathbb{R}_0^+$ ili $\mathcal{D}: x\in[0,+\infty)$ ili $\mathcal{D}: x\geq 0$

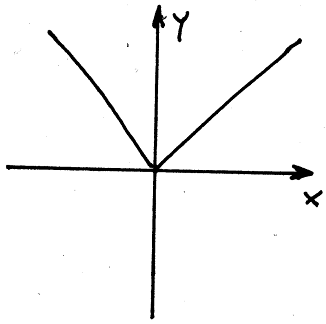
c) $Y=\log x$, Rj. $\mathcal{D}: x\in\mathbb{R}^+$ ili $\mathcal{D}: x\in(0,+\infty)$ ili $\mathcal{D}: x>0$

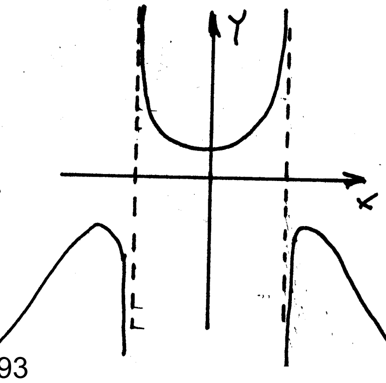
d) $Y=\frac{1}{\sqrt{x}}$, Rj. $\mathcal{D}: x\in\mathbb{R}^+$ ili $\mathcal{D}: x\in(0,+\infty)$ ili $\mathcal{D}: x>0$

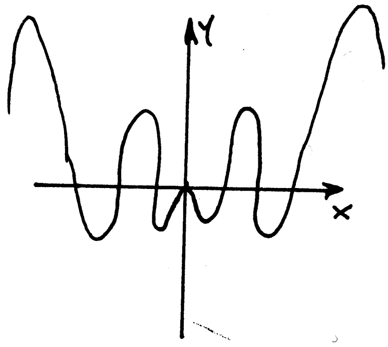
e) $Y=\frac{\log x}{x-2}$, Rj. $x>0$, $x-2\neq 0$, $\mathcal{D}: x\in\mathbb{R}^+\setminus\{2\}$ ili $\mathcal{D}: x\in(0,2)\cup(2,+\infty)$

F-ja je parna ako je $\forall(x\in\mathcal{D})\ f(-x)=f(x)$. Grafik parne f-je je simetričan u odnosu na Y-osu i f-ju je dovoljno ispitati za $x\geq 0$. Grafici parnih f-ja:

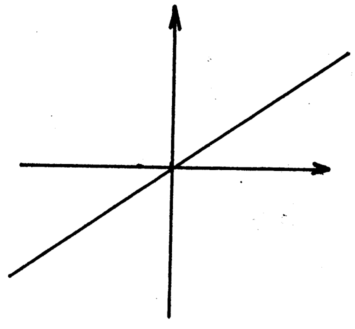


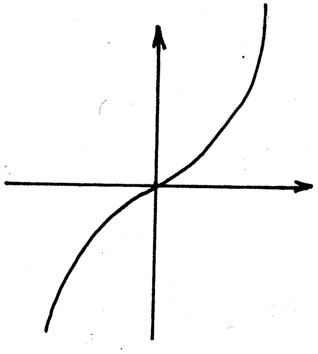


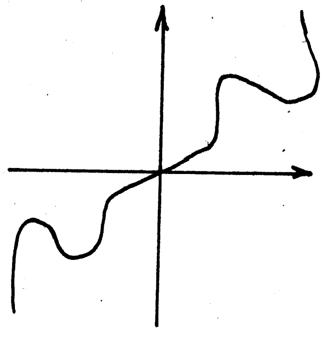


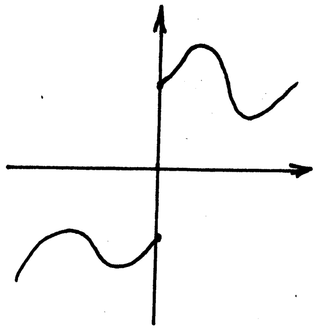


Ako je $\forall (x \in D)$ $f(-x)=-f(x)$ f-ja $f(x)$ je neparna f-ja.
Grafik neparne f-je je simetričan u odnosu na koordinatni početak $(0,0)$ pa je f-ju dovoljno ispitati za $x \geq 0$.
Grafici neparnih f-ja:









② Odrediti parnost i neparnost sljedećih f-ja

a) $y=\frac{x^3}{x^2-4}$ Rj. $f(-x)=\frac{(-x)^3}{(-x)^2-4}=\frac{-x^3}{x^2-4}=-\frac{x^3}{x^2-4}=-f(x)$

f-ja je neparna

b) $y=\frac{x^2+1}{\sqrt{x^2-1}}$ Rj. $f(-x)=\frac{(-x)^2+1}{\sqrt{(-x)^2-1}}=\frac{x^2+1}{\sqrt{x^2-1}}=f(x)$ f-ja f(x) je parna

c) $y=\frac{(x+1)^3}{(x-1)^2}$ Rj. Parnost i neparnost ima smisla ispitati samo ako je D simetrično. U našem slučaju $D:(-\infty,1)\cup(1,+\infty)$ nije simetrično pa f-ja nije ni parna ni neparna.

II način: $f(-x)=\frac{(-x+1)^3}{(-x-1)^2}$ $\Rightarrow$ f-ja nije ni parna ni neparna

---

Neka je data f-ja $y=f(x)$.
Ako je za svako $x \in (a,b)$ $y'(x)<0$ tada f-ja y opada ($\searrow$) na (a,b)
Ako je za svako $x \in (a,b)$ $y'(x)>0$ tada f-ja y $\nearrow$ (raste) na (a,b)
Rješenjem jednačine $y'=0$ dobijamo stacionarne tačke $x_1, x_2, \dots, x_n$ koje konkurišu za ekstrem. Stacionarne tačke $x_1, x_2, \dots, x_n$ mogu ali i ne moraju da budu tačke u kojima f-ja poprima ekstrem. Da li je stacionarna tačka x_1 ekstrem možemo zaključiti na dva načina:

I način: Na osnovu tabele rasta i opadanja, $\nearrow \overset{x_1}{\ } \searrow$ MAX $\searrow \overset{x_1}{\ } \nearrow$ MIN

II način: x_1 je stacionarna tačka

Ako je $y''(x_1)<0 \Rightarrow (x_1, f(x_1))$ je tačka u kojoj f-ja y ima maksimalnu vrijednost

Ako je $y''(x_1)>0 \Rightarrow (x_1, f(x_1))$ je tačka u kojoj f-ja y ima minimalnu vrijednost

③ Nađi ekstreme i intervale rasta i opadanja sljedećih f-ja: a) $y=\frac{x^3}{x^2-4}$ Rj. $D: x\in(-\infty,-2)\cup(-2,2)\cup(2,+\infty)$

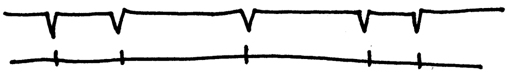
$$y'=\left(\frac{x^3}{x^2-4}\right)'=\frac{3x^2(x^2-4)-x^3\cdot 2x}{(x^2-4)^2}=\frac{x^2(3x^2-12-2x^2)}{(x^2-4)^2}=\frac{x^2(x^2-12)}{(x^2-4)^2}$$

$y''=0$ ako i samo ako $x^2=0$ ili $x^2-12=0$

$x=0$ ili $x_{1,2}=\pm\sqrt{12}$ tj. $x_{1,2}=\pm 2\sqrt{3}$

Stacionarne tačke su $x_1=0$, $x_2=-2\sqrt{3}$ i $x_3=2\sqrt{3}$.

prekidi f-je y + nule f-je y'



rast i opadanje

x	$(-\infty,-2\sqrt{3})$	$(-2\sqrt{3},-2)$	$(-2,0)$	$(0,2)$	$(2,2\sqrt{3})$	$(2\sqrt{3},+\infty)$
y'	+	−	−	−	−	+
y	↗	↘	↘	↘	↘	↗

MAX MIN

$$f(-2\sqrt{3})=\frac{(-2\sqrt{3})^3}{(-2\sqrt{3})^2-4}=\frac{-24\sqrt{3}}{8}=-3\sqrt{3}$$

$$f(2\sqrt{3})=\frac{24\sqrt{3}}{12-4}=3\sqrt{3}$$

$M(-2\sqrt{3},-3\sqrt{3})$ je tačka lokalnog maksimuma a tačka $N(2\sqrt{3},3\sqrt{3})$ je tačka lokalnog minimuma

b) $y=\frac{x^2+1}{\sqrt{x^2-1}}$ Rj. $D: x\in(-\infty,-1)\cup(1,+\infty)$

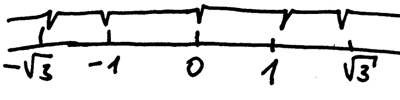
$$y'=\frac{2x\sqrt{x^2-1}-(x^2+1)\cdot\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-1}}}{x^2-1}=\frac{2x(x^2-1)-x(x^2+1)}{(x^2-1)\sqrt{x^2-1}}=\frac{x(2x^2-2-x^2-1)}{(x^2-1)\sqrt{x^2-1}}$$

$y'=\frac{x(x^2-3)}{(x^2-1)\sqrt{x^2-1}}$, $y'=0$ akko $x=0$ ili $x^2-3=0$

$x_{1,2}=\pm\sqrt{3}$

Stacionarne tačke su $x_1=0$, $x_2=-\sqrt{3}$ ili $x_3=\sqrt{3}$.

prekidi f-je y + nule f-je y'



x	$(-\infty,-\sqrt{2})$	$(-\sqrt{3},-1)$	$(-1,0)$	$(0,1)$	$(1,\sqrt{3})$	$(\sqrt{3},+\infty)$
y'	−	+	▨	▨	−	+
y	↘	↗	▨	▨	↘	↗

MIN MIN

rast i opadanje

Tačke $M(-\sqrt{3},2\sqrt{2})$ i $N(\sqrt{3},2\sqrt{2})$ su tačke lokalnog minimuma.

$f(-\sqrt{3})=\frac{3+1}{\sqrt{3-1}}=\frac{4}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=2\sqrt{2}$, $f(\sqrt{3})=2\sqrt{2}$

(4.) Ispitati i grafički predstaviti f-ju $Y=\frac{x}{x-3}$.

Rj. definiciono područje

$x-3\neq 0$

$x\neq 3$

$\mathcal{D}: (-\infty,3)\cup(3,+\infty)$

parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$

$\Rightarrow$ f-ja nije ni parna ni neparna.

F-ja f(x) je periodična sa periodom T ako $f(x+T)=f(x)$.
Periodične su samo trigonometrijske f-je

F-ja nije periodična

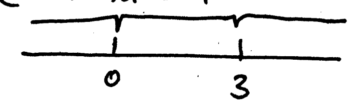
nule, presjek sa Y-osom, znak f-je

tačka oblika (A, 0) je nula f-je, a tačka oblika (0, B) je tačka presjeka sa Y-osom.

$f(x)=\frac{x}{x-3}$, $f(0)=\frac{0}{-3}=0$

(0,0) je nula f-je i presjek sa Y-osom

prelazi f-je i nule Y



x	$(-\infty,0)$	$(0,3)$	$(3,+\infty)$
x	−	+	+
x−3	−	−	+
Y	+	−	+

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

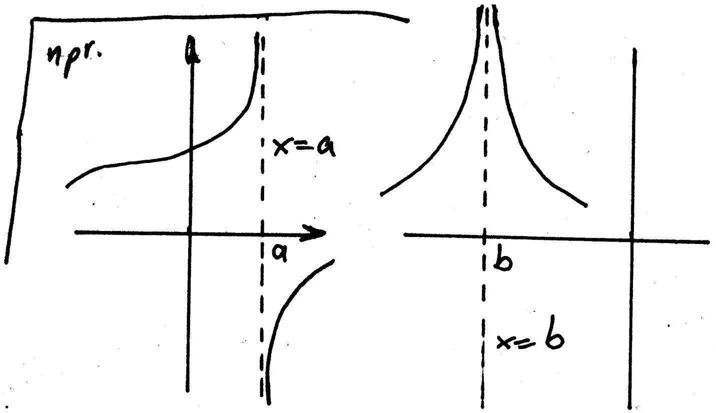
Neka je a tačka u kojoj f-ja nije definisana.

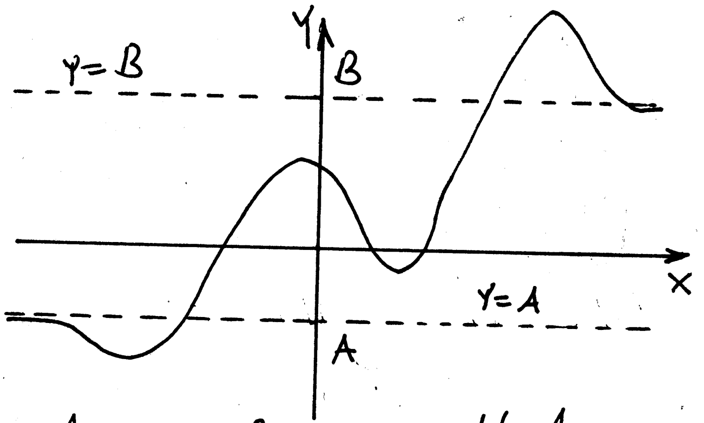
$\lim_{x\to a-0} f(x)=-\infty$ (ili $+\infty$) $\Rightarrow$ $x=a$ je vertikalna asimptota

$\lim_{x\to a+0} f(x)=+\infty$ (ili $-\infty$) $\Rightarrow$ $x=a$ je vertikalna asimptota

$\lim_{x\to\infty} f(x)=A$, $A\neq+\infty$ i $A\neq-\infty$ $\Rightarrow$ $Y=A$ je horizontalna asimptota

$\lim_{x\to-\infty} f(x)=B$, $B\neq+\infty$ i $B\neq-\infty$ $\Rightarrow$ $Y=B$ je horizontalna asimptota



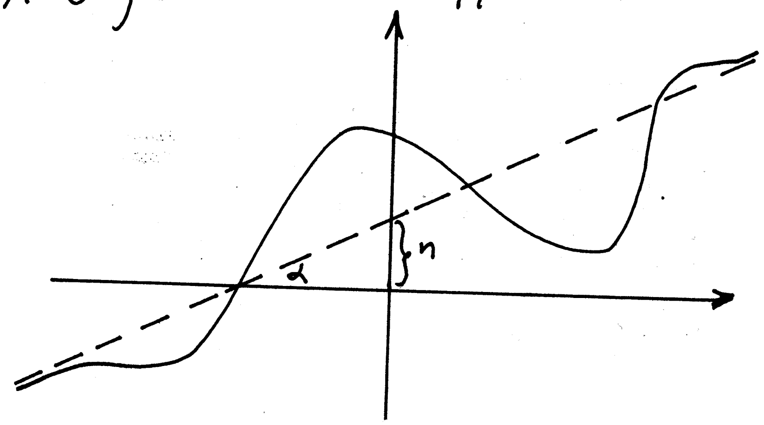


$x=a$ i $x=b$ su V.A. $y=A$ i $y=B$ su H.A.

Ako f-ja nema horizontalnu asimptotu onda tražimo kosu asimptotu u obliku $y=kx+n$.

$$k=\lim_{x\to\infty}\frac{f(x)}{x},\quad n=\lim_{x\to\infty}\left[f(x)-kx\right]$$

Ako je $k=\pm\infty$ ili $k=0$ f-ja nema kosu asimptotu.



U beskonačnosti f-ja ne dodiruje asimptotu ali je u "normalnom" položaju u nekoj tački može sijeći.

Za $x=3$ f-ja nije definisana

$$\lim_{x\to 3-0}\frac{x}{x-3}=\frac{3-0}{3-0-3}=\frac{3-0}{-0}=-\infty$$

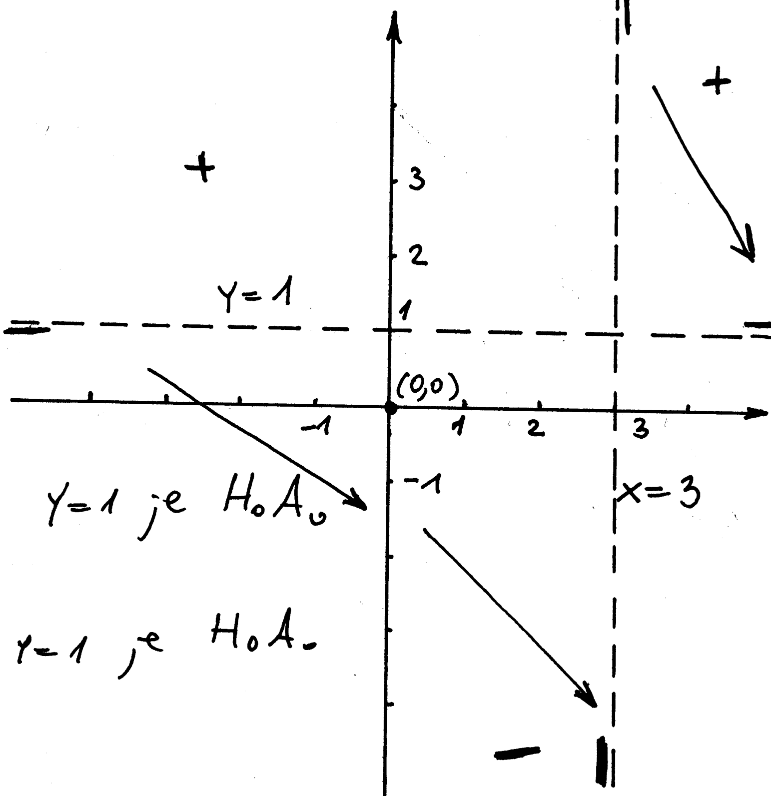
$\Rightarrow$ $x=3$ je V.A. (sa lijeve str.)

$$\lim_{x\to 3+0}\frac{x}{x-3}=\frac{3+0}{3+0-3}=\frac{3+0}{+0}=+\infty$$

$\Rightarrow$ $x=3$ je V.A. (sa desne str.)

$$\lim_{x\to+\infty}\frac{x\ /:x}{x-3\ /:x}=\lim_{x\to+\infty}\frac{1}{1-\underbrace{\frac{3}{x}}_{\to 0}}=1 \Rightarrow y=1 \text{ je H.A.}$$

$$\lim_{x\to-\infty}\frac{x}{x-3}=\lim_{x\to-\infty}\frac{1}{1-\frac{3}{x}}=1 \Rightarrow y=1 \text{ je H.A.}$$



F-ja nema kosu asimptotu.

Poslije ovog koraka počinjemo sa skiciranjem grafa f-je.

intervali rasta i opadanja

$$Y' = \left(\frac{x}{x-3}\right)' = \frac{1\cdot(x-3) - x\cdot 1}{x-3} = \frac{-3}{(x-3)^2} < 0 \quad \forall x \in D$$

f-ja Y ↘ za ∀(x∈D)

ekstremi f-je

$$Y' = 0, \quad Y' = \frac{-3}{(x-3)^2} \neq 0 \quad \forall x \in D \Rightarrow$$ f-ja nema ekstrema

prevojne tačke i intervali konveksnosti i konkavnosti

Konveksnost (∪) i konkavnost (∩) f-je određujemo na osnovu znaka f-je y''.

Ako je $\forall x \in (a,b)$ $Y''(x) < 0 \Rightarrow$ f-ja y je ∩ na (a,b)

Ako je $\forall x \in (a,b)$ $Y''(x) > 0 \Rightarrow$ f-ja Y je ∪ na (a,b)

Za $Y'' = 0$ dobijemo tačke $x_1, x_2, \ldots, x_n$ koje konkurišu za prevojne tačke. Tačka x_1 je prevojna tačka ako u njoj f-ja Y prelazi iz ∪ u ∩ i obrnuto

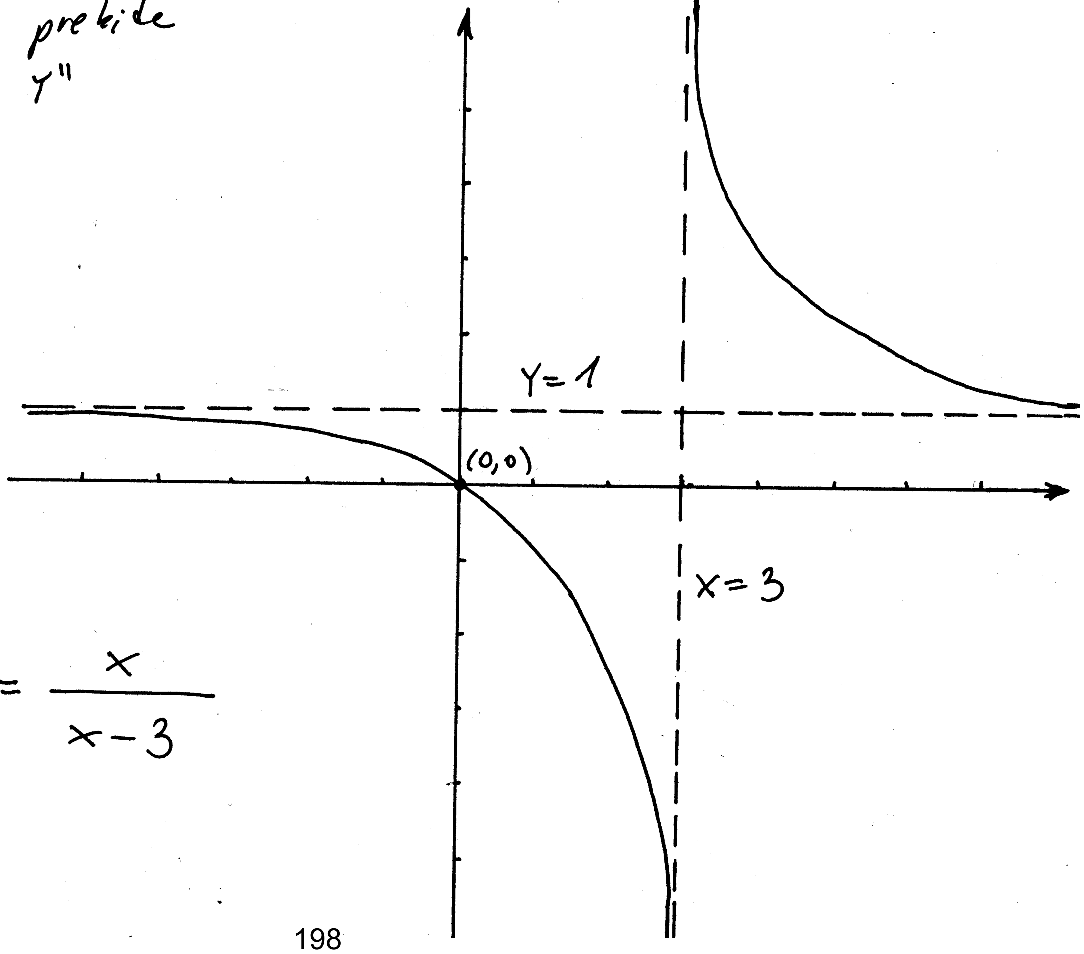
$$Y'' = \left(\frac{-3}{(x-3)^2}\right)' = \left(-3(x-3)^{-2}\right)' = 6(x-3)^{-3}\cdot 1 = \frac{6}{(x-3)^3} \neq 0 \Rightarrow$$ f-ja nema prevojnih tački

u tabelu stavljamo prekide f-je Y + nule f-je y''

x	$(-\infty, 3)$	$(3, +\infty)$
Y''	—	+
Y	∩	∪

konveksnost i konkavnost

grafik f-je



$$Y = \frac{x}{x-3}$$

(#) Ispitati f-ju i nacrtati joj grafik $y = \frac{3x}{1+x^3}$.

Rj. definiciono područje

$1 + x^3 \neq 0$

$x^3 \neq -1$

$x \neq -1$

$D: x \in (-\infty, -1) \cup (-1, +\infty)$

parnost, neparnost, periodičnost

$f(-x) = \frac{3 \cdot (-x)}{1 + (-x^3)} = -\frac{3x}{1 - x^3}$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$Y = 0$

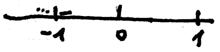
$\frac{3x}{1+x^3} = 0$

$x = 0$

(0,0) je nula f-je i presjek sa Y-osom

x	$(-\infty,-1)$	$(-1,0)$	$(0,+\infty)$
$3x$	−	−	+
$1+x^3$	−	+	+
Y	+	−	+

znak f-je



ponašanje na krajevima intervala definisanosti i asimptote

za vrijednost x = -1 f-ja ima prekid

$\lim_{x \to -1-0} f(x) = \lim_{x \to -1-0} \frac{3x}{1+x^3} = \frac{3(-1-0)}{1+(-1-0)^3} = \frac{3(-1-0)}{1-1-0} = \frac{-3-0}{-0} = +\infty \Rightarrow x = -1$ je V.A.

$\lim_{x \to -1+0} f(x) = \lim_{x \to -1+0} \frac{3x}{1+x^3} = \frac{3(-1+0)}{1+(-1+0)^3} = \frac{-3+0}{1-1+0} = \frac{-3+0}{+0} = -\infty \Rightarrow x = -1$ je V.A.

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{3x \,:x}{1+x^3 \,:x} = \lim_{x \to -\infty} \frac{3}{\underbrace{\frac{1}{x}}_{\to 0} + x^2} = 0 \Rightarrow Y = 0$ je H.A.

$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{3}{\underbrace{\frac{1}{x}}_{\to 0} + x^2} = 0 \Rightarrow Y = 0$ je H.A. f-ja nema K.A.

rast i opadanje

$Y' = \left(\frac{3x}{1+x^3}\right)' = 3 \cdot \frac{1 \cdot (1+x^3) - x \cdot 3x^2}{(1+x^3)^2} = 3 \frac{1 + x^3 - 3x^3}{(1+x^3)^2}$

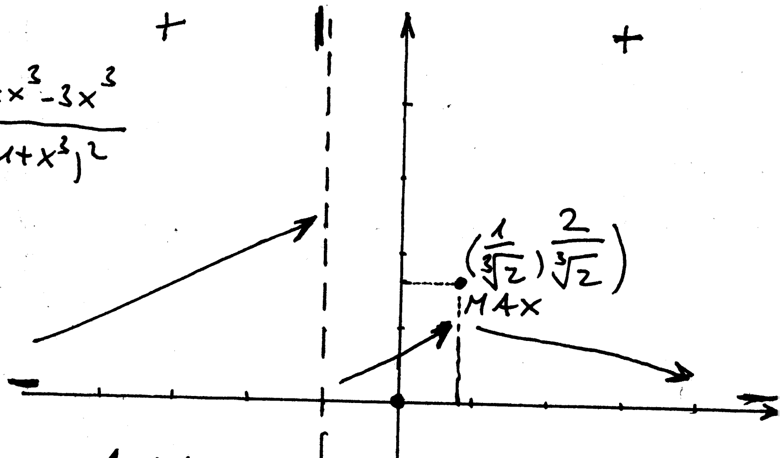
$Y' = 3 \cdot \frac{1 - 2x^3}{(1+x^3)^2}$

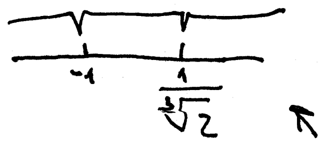
$y' = 0$ akko $1 - 2x^3 = 0$

$2x^3 = 1$

$x^3 = \frac{1}{2}$

$x = \frac{1}{\sqrt[3]{2}} \approx 0{,}8$





↖ prekidi Y + nule Y'

X	$(-\infty,-1)$	$(-1,\frac{1}{\sqrt[3]{2}})$	$(\frac{1}{\sqrt[3]{2}},+\infty)$
Y'	+	+	−
Y	↗	↗	↘

MAX

ekstrema f-je

Na osnovu tabele

$f\left(\frac{1}{\sqrt[3]{2}}\right) = \frac{\frac{3}{\sqrt[3]{2}}}{1 + \frac{1}{2}} = \frac{\frac{3}{\sqrt[3]{2}}}{\frac{3}{2}} = \frac{2}{\sqrt[3]{2}} \approx 1{,}6$

$\Rightarrow$ $\left(\frac{1}{\sqrt[3]{2}}, \frac{2}{\sqrt[3]{2}}\right)$ je tačka maksimuma prevojne tačke i intervali konveksnosti i konkavnosti

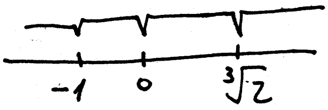
$$y' = 3 \cdot \frac{1-2x^3}{(1+x^3)^2} \quad , \quad y'' = 3 \cdot \frac{-6x^2 \cdot (1+x^3)^{\cancel{2}} - (1-2x^3)\cdot 2\cancel{(1+x^3)}\cdot 3x^2}{(1+x^3)^3 \cdot \cancel{(1+x^3)}} =$$

$$= 3 \cdot \frac{-6x^2 - 6x^5 - 6x^2 + 12x^5}{(1+x^3)^3} = 3 \cdot \frac{6x^5 - 12x^2}{(1+x^3)^3}$$

$$y'' = 18 \cdot \frac{x^5 - 2x^2}{(1+x^3)^3} = \frac{18x^2(x^3-2)}{(1+x^3)^3}$$

$y''=0$ akko $x=0$ ili $x^3-2=0$

$x_1=0$ $\qquad x_2 = \sqrt[3]{2} \approx 1{,}3$



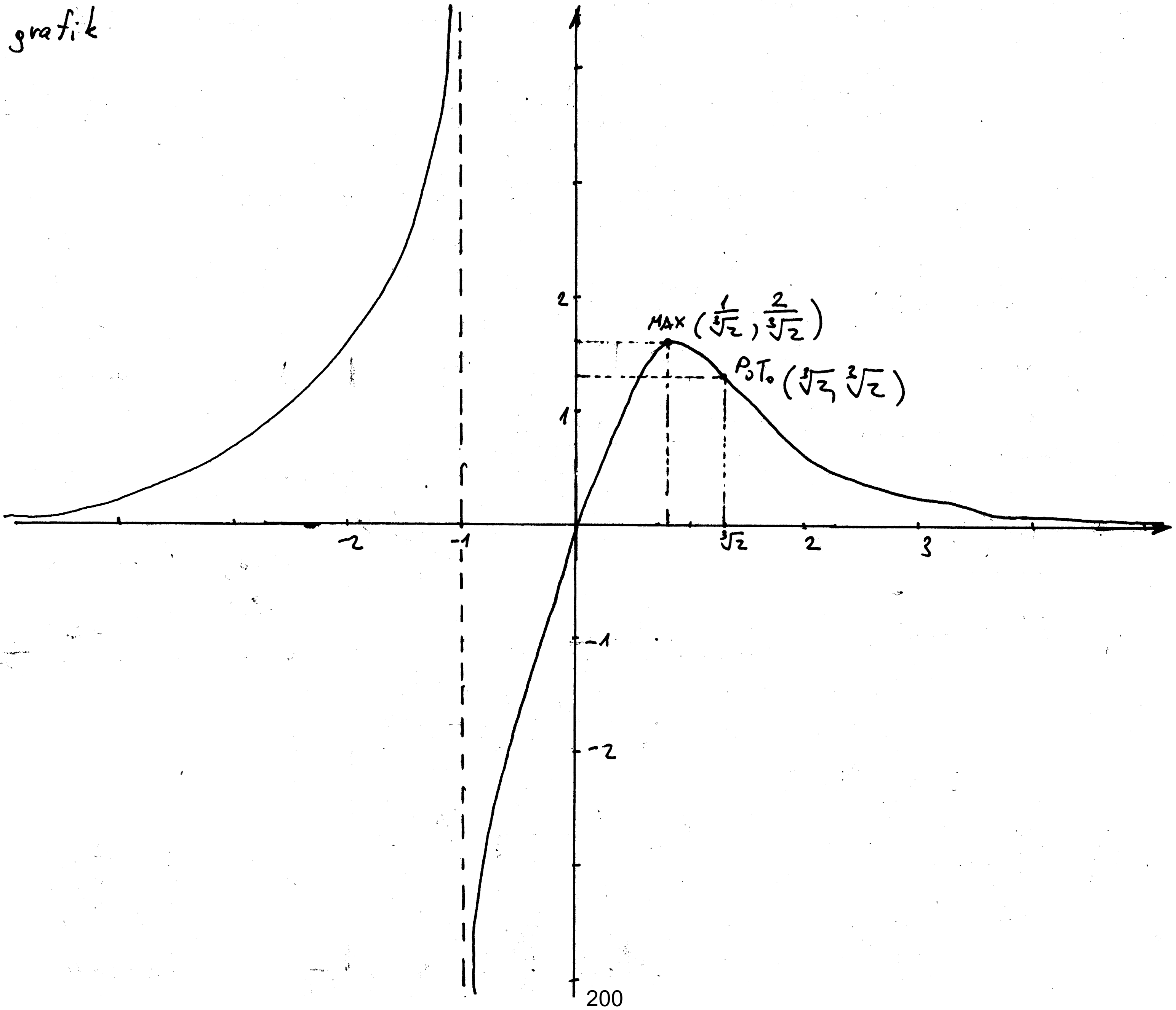
x	$(-\infty,-1)$	$(-1,0)$	$(0,\sqrt[3]{2})$	$(\sqrt[3]{2},+\infty)$
y''	+	−	−	+
y	∪	∩	∩	∪

P_0T_0

$$f(\sqrt[3]{2}) = \frac{3\sqrt[3]{2}}{1+2} = \sqrt[3]{2}$$

$(\sqrt[3]{2}, \sqrt[3]{2})$ je prevojna tačka

grafik



(#) Ispitati f-ju i nacrtati joj grafik $y=\frac{(2x-1)^3}{(x+2)^2}$.

Rj. definiciono područje

$D: x\in R\setminus\{-2\}$

parnost, neparnost, periodičnost

D nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y=0$ akko $(2x-1)^3=0$

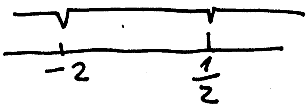
$2x-1=0$

$x=\frac{1}{2}$

$(\frac{1}{2},0)$ je nula f-je

$f(0)=\frac{(-1)^3}{2^2}=-\frac{1}{4}$

$(0,-\frac{1}{4})$ je tačka presjeka sa y-osom



x	$(-\infty,-2)$	$(-2,\frac{1}{2})$	$(\frac{1}{2},+\infty)$
$(2x-1)^3$	$-$	$-$	● $+$
y	$-$	$-$	$+$

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

za $x=-2$ f-ja ima prekid

$$\lim_{x\to-2-0}f(x)=\lim_{x\to-2-0}\frac{(2x-1)^3}{(x+2)^2}=\frac{(2\cdot(-2-0)-1)^3}{(-2-0+2)^2}=\frac{(-5-0)^3}{+0}=-\infty \Rightarrow x=-2 \text{ je V. A.}$$

(sa lijeve strane)

$$\lim_{x\to-2+0}f(x)=\lim_{x\to-2+0}\frac{(2x-1)^3}{(x+2)^2}=\frac{(2\cdot(-2+0)-1)^3}{(-2+0+2)^2}=\frac{(-5-0)^3}{+0}=-\infty \Rightarrow x=-2 \text{ je V. A.}$$

(sa desne strane)

$$(2x-1)^3=(2x)^3+3\cdot(2x)^2\cdot(-1)+3\cdot 2x\cdot(-1)^2+(-1)^3=8x^3-12x^2+6x-1$$

1
1 1
1 2 1
1 3 3 1

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{(2x-1)^3}{(x+2)^2}=\lim_{x\to-\infty}\frac{8x^3-12x^2+6x-1 \quad /:x^3}{x^2+4x+2 \quad /:x^3}=\lim_{x\to-\infty}\frac{8-\frac{12}{x}+\frac{6}{x^2}-\frac{1}{x^3}}{\frac{1}{x}+\frac{4}{x^2}+\frac{2}{x^3}}=-\infty$$

$$\lim_{x\to+\infty}f(x)=\lim_{x\to+\infty}\frac{8x^3-12x^2+6x-1 \quad /:x^2}{x^2+4x+2 \quad /:x^2}=\lim_{x\to+\infty}\frac{8x-12+\frac{6}{x}-\frac{1}{x^2}}{1+\frac{4}{x}+\frac{2}{x^2}}=+\infty$$

f-ja nema H. A.

kosa asimptota je oblika $y=kx+n$

$$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{8x^3-12x^2+6x-1 \quad /:x^3}{x^3+4x^2+2x \quad /:x^3}$$

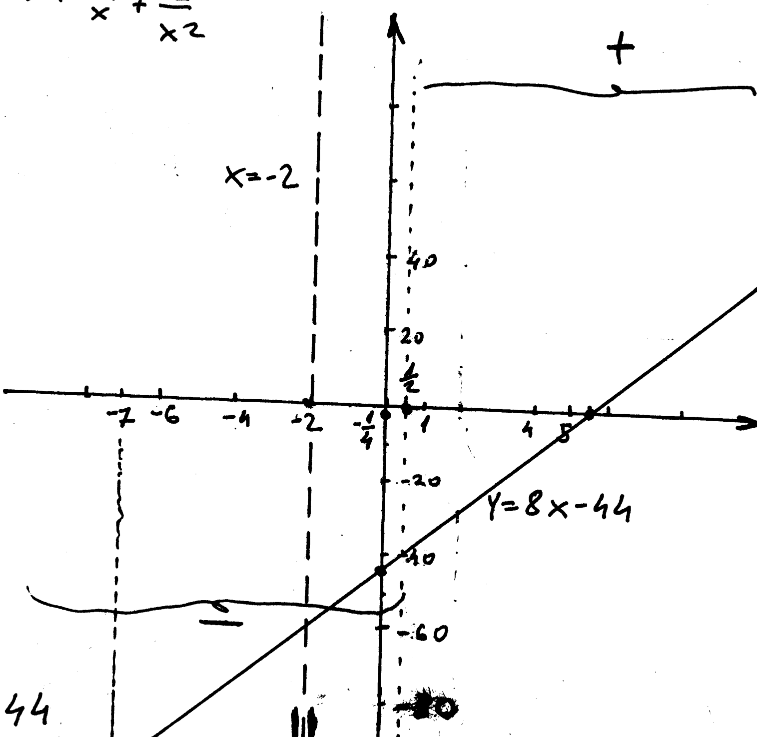
$$=\lim_{x\to\infty}\frac{8-\frac{12}{x}+\frac{8}{x^2}-\frac{1}{x^3}}{1+\frac{4}{x}+\frac{2}{x^2}}=8$$

$$n=\lim_{x\to\infty}[f(x)-k\cdot x]=\lim_{x\to\infty}\left(\frac{(2x-1)^3}{(x+2)^2}-8x\right)=$$

$$=\lim_{x\to\infty}\frac{8x^3-12x^2+6x-1-8x(x^2+4x+2)}{(x+2)^2}=$$

$$=\lim_{x\to\infty}\frac{\cancel{8x^3}-12x^2+6x-1-\cancel{8x^3}-32x^2-16x}{x^2+4x+4}$$

$$=\lim_{x\to\infty}\frac{-44x^2-10x-1 \quad :x^2}{x^2+4x+4 \quad :x^2}=\lim_{x\to\infty}\frac{-44-\frac{10}{x}-\frac{1}{x^2}}{1+\frac{4}{x}+\frac{4}{x^2}}=-44$$



$Y = 8x - 44$ je $K_o A_o$ (počinjemo sa skiciranjem grafa)

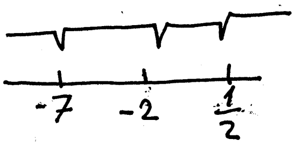
($Y = 8x - 44$, $Y = 0 \Rightarrow 8x = 44$ $\quad x = 0 \Rightarrow Y = -44$)

$$x = \frac{44}{8} = \frac{11}{2} = 5{,}5$$

rast i opadanje

$$Y' = \left(\frac{(2x-1)^3}{(x+2)^2}\right)' = \frac{3(2x-1)^2 \cdot 2\,(x+2)^{\cancel{2}\,1} - (2x-1)^3 \cancel{2(x+2)}}{(x+2)^{\cancel{4}\,3}} = \frac{2(2x-1)^2(3x+6-2x+1)}{(x+2)^3} = \frac{2(2x-1)^2(x+7)}{(x+2)^3}$$

$Y' = 0$ akko $x = \frac{1}{2}$ i $x = -7$



← nule Y' + prekidi od Y

x	$(-\infty, -7)$	$(-7, -2)$	$(-2, \frac{1}{2})$	$(\frac{1}{2}, +\infty)$
Y'	+	−	+	+
Y	↗	↘	↗	↗
	max			

rast i opadanje

$$f(-7) = \frac{(-15)^3}{(-5)^2} = \frac{-3375}{25} = -135$$

ekstremi f-je Na osnovu tabele rasta i opadanja $M(-7, -135)$ je tačka maksimuma

prevojne tačke i intervali konveksnosti i konkavnosti

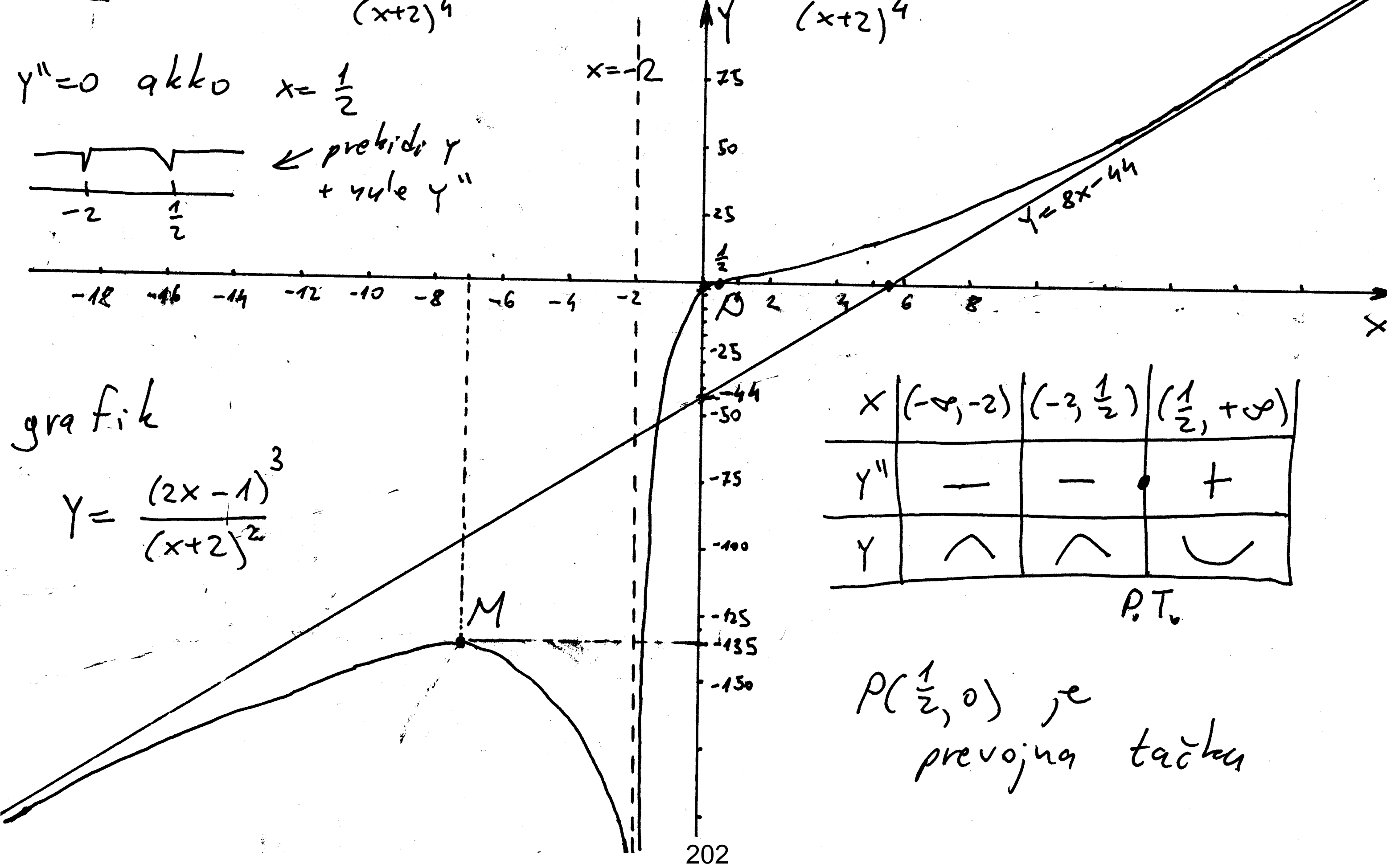
$$Y'' = \left(2\,\frac{(2x-1)^2(x+7)}{(x+2)^3}\right)' = 2 \cdot \frac{\left[2(2x-1)\cdot 2\cdot(x+7) + (2x-1)^2\right](x+2)^{\cancel{3}\,1} - (2x-1)^2(x+7)\,3\cancel{(x+2)}^{2}}{(x+2)^{\cancel{6}\,4}} =$$

$$= 2 \cdot \frac{\left[(2x-1)(4x+28+2x-1)\right](x+2) - 3(2x-1)^2(x+7)}{(x+2)^4} = 2 \cdot \frac{(2x-1)\left[\overset{6x^2+12x+27x+54}{(6x+27)(x+2)} - 3\overset{2x^2+13x-7}{(2x-1)(x+7)}\right]}{(x+2)^4}$$

$$Y'' = 2 \cdot \frac{(2x-1)(\cancel{6x^2}+39x+54-\cancel{6x^2}-39x+21)}{(x+2)^4} = \frac{150(2x-1)}{(x+2)^4}$$

$Y'' = 0$ akko $x = \frac{1}{2}$

← prekidi Y + nule Y''



grafik

$$Y = \frac{(2x-1)^3}{(x+2)^2}$$

X	$(-\infty, -2)$	$(-2, \frac{1}{2})$	$(\frac{1}{2}, +\infty)$
Y''	−	−	+
Y	⌒	⌒	⌣
		P.T.	

$P(\frac{1}{2}, 0)$ je prevojna tačka

(#) Ispitati i grafički predstaviti f-ju $Y = \dfrac{x^2+5x}{x^2+2x+1}$.

Rj. definiciono područje

$x^2+2x+1 \neq 0$

$D = 4-4=0$

$(x+1)^2 \neq 0$

$x \neq -1$

$D: x \in \mathbb{R} \setminus \{-1\}$

parnost, neparnost, periodičnost

D nije simetrično ⇒

f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa Y-osom, znak f-je

$Y=0$ akko $x^2+5x=0$

$x(x+5)=0$

$x_1=0$ ili $x_2=-5$

$(0,0)$ i $(-5,0)$ su nule f-je

$(0,0)$ je tačka presjeka sa Y-osom.

$Y = \dfrac{x(x+5)}{(x+1)^2}$ $-5 \quad -1 \quad 0$

x	$(-\infty,-5)$	$(-5,-1)$	$(-1,0)$	$(0,+\infty)$
x	$-$	$-$	$-$	$+$
$x+5$	$-$	$+$	$+$	$+$
Y	$+$	$-$	$-$	$+$

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

za $x=-1$ f-ja ima prekid.

$$\lim_{x\to -1-0} f(x) = \lim_{x\to -1-0} \frac{x(x+5)}{(x+1)^2} = \frac{(-1-0)(-1-0+5)}{(-1-0+1)^2} = \frac{(-1-0)(4-0)}{+0} = -\infty \Rightarrow x=-1 \text{ je } V_oA_o$$

$$\lim_{x\to -1+0} f(x) = \lim_{x\to -1+0} \frac{x(x+5)}{(x+1)^2} = \frac{(-1+0)(-1+0+5)}{(-1+0+1)^2} = \frac{(-1+0)(4+0)}{+0} = -\infty \Rightarrow x=-1 \text{ je } V_oA_o$$

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{x^2+5x \; /:x^2}{x^2+2x+1 \; /:x^2} = \lim_{x\to -\infty} \frac{1+\frac{5}{x}}{1+\frac{2}{x}+\frac{1}{x^2}} = 1 \Rightarrow Y=1 \text{ je } H_oA_o$$

isto vrijedi i za $\displaystyle\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{1+\frac{5}{x}}{1+\frac{2}{x}+\frac{1}{x^2}} = 1 \Rightarrow Y=1$ je H_oA_o

nakon ovog koraka počinjemo skicirati graf.

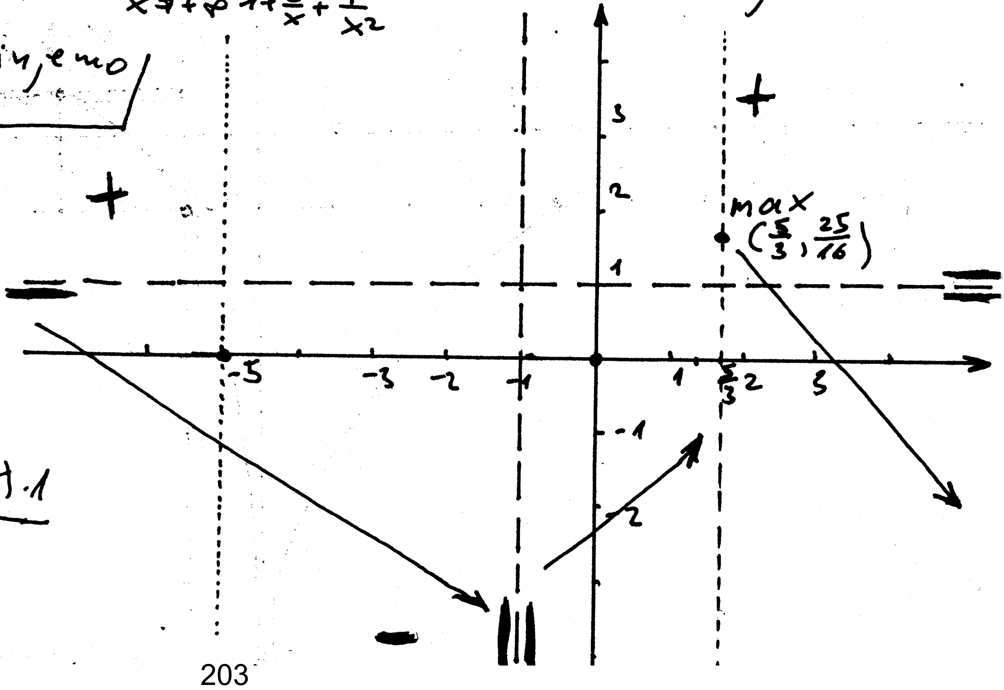
f-ja nema K_oA_o

rast i opadanje

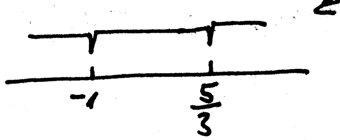
$$Y' = \left(\frac{x^2+5x}{(x+1)^2}\right)' =$$

$$= \frac{(2x+5)(x+1)^{\cancel{2}\,1} - (x^2+5x)\,2\cancel{(x+1)}\cdot 1}{(x+1)^{\cancel{4}\,3}}$$

$$= \frac{\cancel{2x^2}+5x+2x+5-\cancel{2x^2}-10x}{(x+1)^3}$$



$$Y' = \frac{-3x+5}{(x+1)^3}$$



← nule Y' + prekidi Y

$Y'=0$ akko $-3x+5=0$

$-3x=-5$

$x=\frac{5}{3}\approx 1{,}6667$

$Y'(-2)=\frac{11}{-1}<0$

X	$(-\infty,-1)$	$(-1,\frac{5}{3})$	$(\frac{5}{3},+\infty)$
Y'	−	+	−
Y	↘	↗	↘

max

rast i opadanje

ekstremi f-je

naosnovu tabele rasta i opadanja f-ja ima maksimum za $x=\frac{5}{3}$

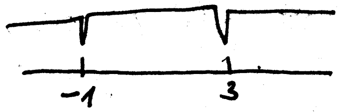
$$f\left(\frac{5}{3}\right)=\frac{\frac{25}{9}+5\cdot\frac{5}{3}}{\left(\frac{5}{3}+1\right)^2}=\frac{\frac{25+25\cdot 3}{9}}{\left(\frac{8}{3}\right)^2}=\frac{\frac{100}{8}:2}{\frac{64}{8}:2}=\frac{50}{32}=\frac{25}{16}\approx 1{,}5625$$

$M\left(\frac{5}{3},\frac{25}{16}\right)$ je tačka maksimuma

prevojne tačke i intervali konveksnosti i konkavnosti

$$Y''=\left(\frac{-3x+5}{(x+1)^3}\right)'=\frac{-3(x+1)^{\overset{1}{\cancel{3}}}-(-3x+5)3\cancel{(x+1)^2}\cdot 1}{(x+1)^4\cdot\cancel{(x+1)^2}}=\frac{-3x-3+9x-15}{(x+1)^4}=\frac{6x-18}{(x+1)^4}$$

$Y''=6\cdot\frac{x-3}{(x+1)^4}$, $Y''=0$ akko $x=3$



← prekidi Y + nule Y''

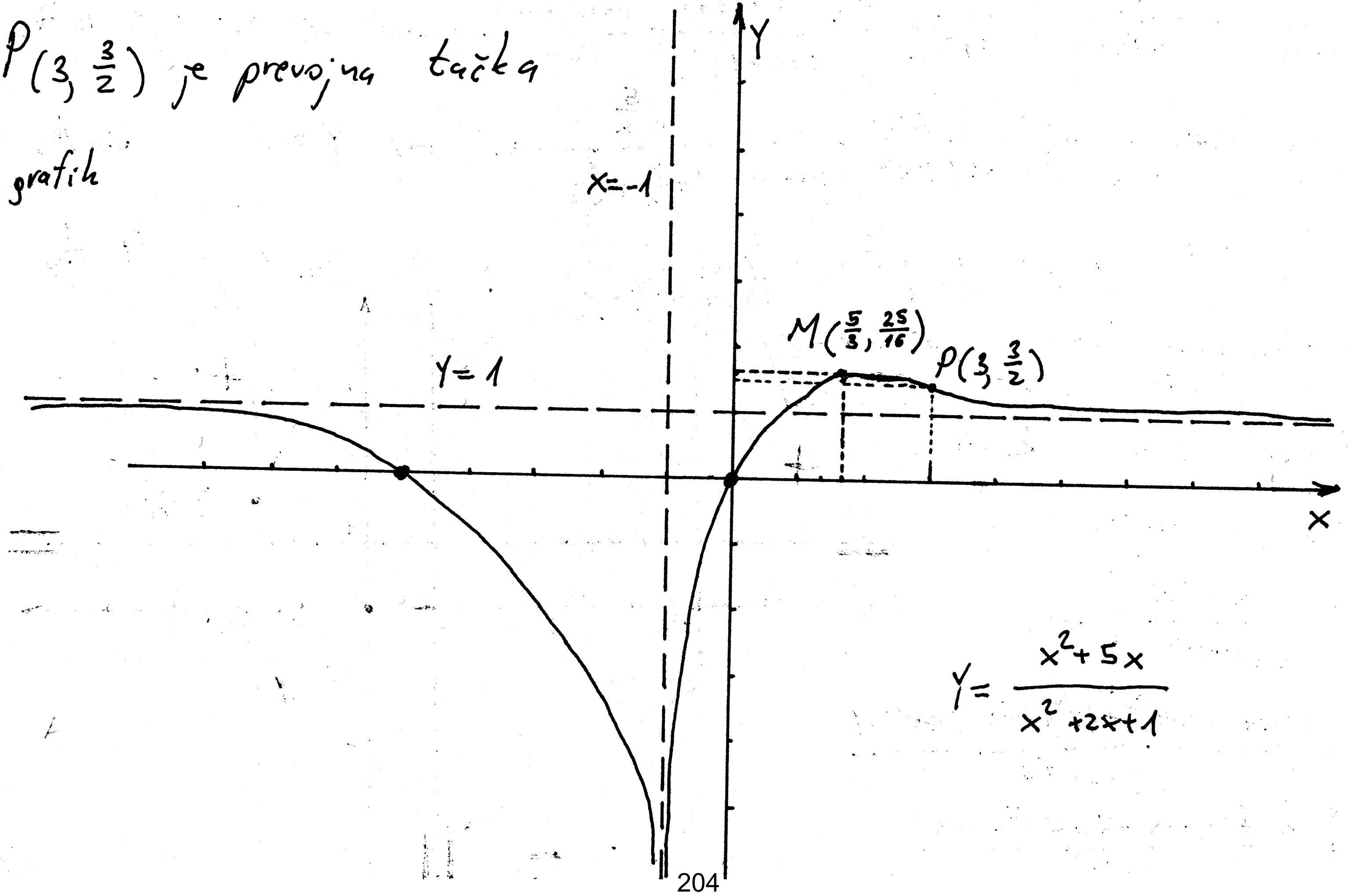
X	$(-\infty,-1)$	$(-1,3)$	$(3,+\infty)$
Y''	−	−	+
Y	⌒	⌒	∪

P_0T_0

$$f(3)=\frac{3^2+5\cdot 3}{(3+1)^2}=\frac{9+15}{16}=\frac{24:4}{16:4}=\frac{6:2}{4:2}=\frac{3}{2}=1{,}5$$

$P\left(3,\frac{3}{2}\right)$ je prevojna tačka

grafik



(#) Odrediti parametre a i b tako da f-ja $y=\frac{x}{x^2+ax+b}$ ima ekstrem u tački $T(2,\frac{1}{7})$. Zatim ispitati tako dobijenu f-ju i nacrtati joj grafik.

Rj

$$f(2)=\frac{1}{7}$$

$$\frac{2}{4+2a+b}=\frac{1}{7}$$

$$4+2a+b=14$$

$$2a+b=10$$

Kandidat za ekstreme su stacionarne tačke

$$y'=\frac{x^2+ax+b-x(2x+a)}{(x^2+ax+b)^2}=\frac{x^2+\cancel{ax}+b-2x^2-\cancel{ax}}{(x^2+ax+b)^2}$$

$$y'=\frac{-x^2+b}{(x^2+ax+b)^2}$$

Potreban uslov da f-ja y ima ekstrem u tački $T(2,\frac{1}{7})$ je $y'(2)=0$.

$$-4+b=0$$

$$b=4$$

$$2a+4=10$$

$$2a=6$$

$$a=3$$

$$y=\frac{x}{x^2+3x+4}$$

definiciono područje

$x^2+3x+4\neq 0$

$D=9-16<0$

$a>0 \quad x^2+3x+4>0 \quad \forall x\in\mathbb{R}$

$\mathcal{D}: x\in\mathbb{R}$

parnost, neparnost, periodičnost

$$f(-x)=\frac{-x}{x^2-3x+4}$$

f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak

$f(x)=0$ akko $x=0$

$(0,0)$ je nula f-je i presjek sa y-osom

x	$(-\infty,0)$	$(0,+\infty)$
y	$-$	$+$

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

f-ja nema prekida $\Rightarrow$ f-ja nema V.A.

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{x}{x^2+3x+4}\,\frac{:x}{:x}=\lim_{x\to\infty}\frac{1}{x+3+\underset{0}{\boxed{\frac{4}{x}}}}=\frac{1}{\infty}=0$$

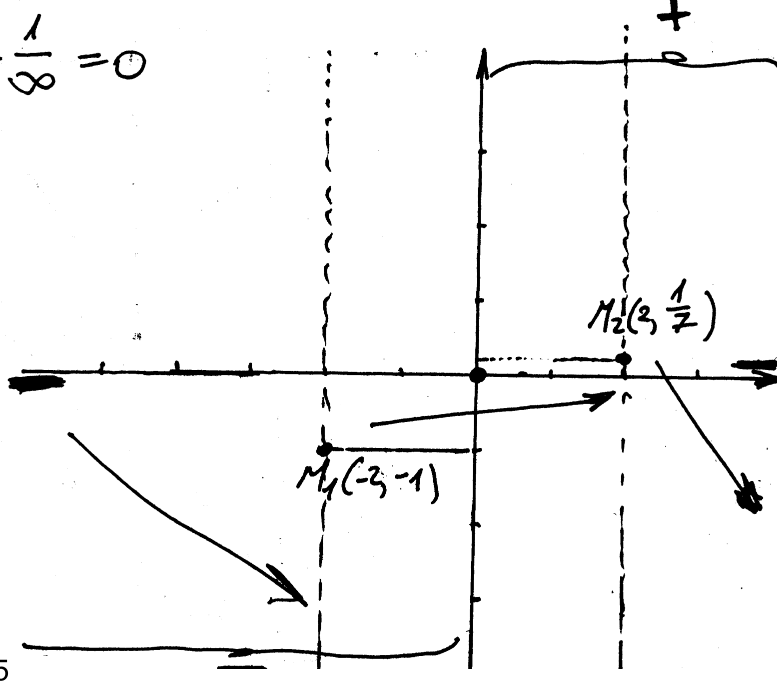
$\Rightarrow y=0$ je H.A.

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{1}{x+3+\frac{4}{x}}=\frac{1}{-\infty}=0$$

$\Rightarrow y=0$ je H.A.

F-ja nema K.A.

Poslije ovog koraka počinjemo skicirati grafik.



rast i opadanje

$$Y' = \frac{-x^2+b}{(x^2+ax+b)^2} \Rightarrow Y' = \frac{4-x^2}{(x^2+3x+4)^2}$$

$Y'=0$ akko $4-x^2=0$

$x_1=-2,\ x_2=2$

$$\frac{-2}{8-6}$$

$$\frac{2}{8+6}=\frac{2}{14}$$

X	$(-\infty,-2)$	$(-2,2)$	$(2,+\infty)$
Y'	−	+	−
Y	↘	↗	↘

rast i opadanje

min $f(-2)=-1$ max $f(2)=\frac{1}{7}$

ekstremi f-je

Na osnovu tabele $M_1(-2,-1)$ je tačka min a $M_2(2\ \frac{1}{7})$ je max.

prevojne tačke i intervali konv. i konk.

$$Y'' = \left(\frac{4-x^2}{(x^2+3x+4)^2}\right)' =$$

$$= \frac{-2x(x^2+3x+4)^{\cancel{2}} - (4-x^2)2\cancel{(x^2+3x+4)}\cdot(2x+3)}{(x^2+3x+4)^{\cancel{4}\,3}} = \frac{-2\left[\underline{x^3}+\cancel{3x^2}+4x+8x+12\underline{-2x^3}-\cancel{3x^2}\right]}{(x^2+3x+4)^3}$$

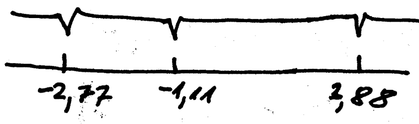
$$Y'' = -2\cdot\frac{-x^3+12x+12}{(x^2+3x+4)^3} = 2\,\frac{x^3-12x-12}{(x^2+3x+4)^3}$$

$Y''=0$ akko $x^3-12x-12=0$

$x_1 \approx 3{,}88$ $x_2 \approx -1{,}11$

$x_3 \approx -2{,}77$

(vrijednosti x_1, x_2 i x_3 su nađene pomoću digitrona koji ima opciju da nađe nule polinoma)



X	$(-\infty,-2{,}77)$	$(-2{,}77,-1{,}11)$	$(-1{,}11,\ 3{,}88)$	$(3{,}88,+\infty)$
Y''	−	+	−	+
Y	∩	∪	∩	∪

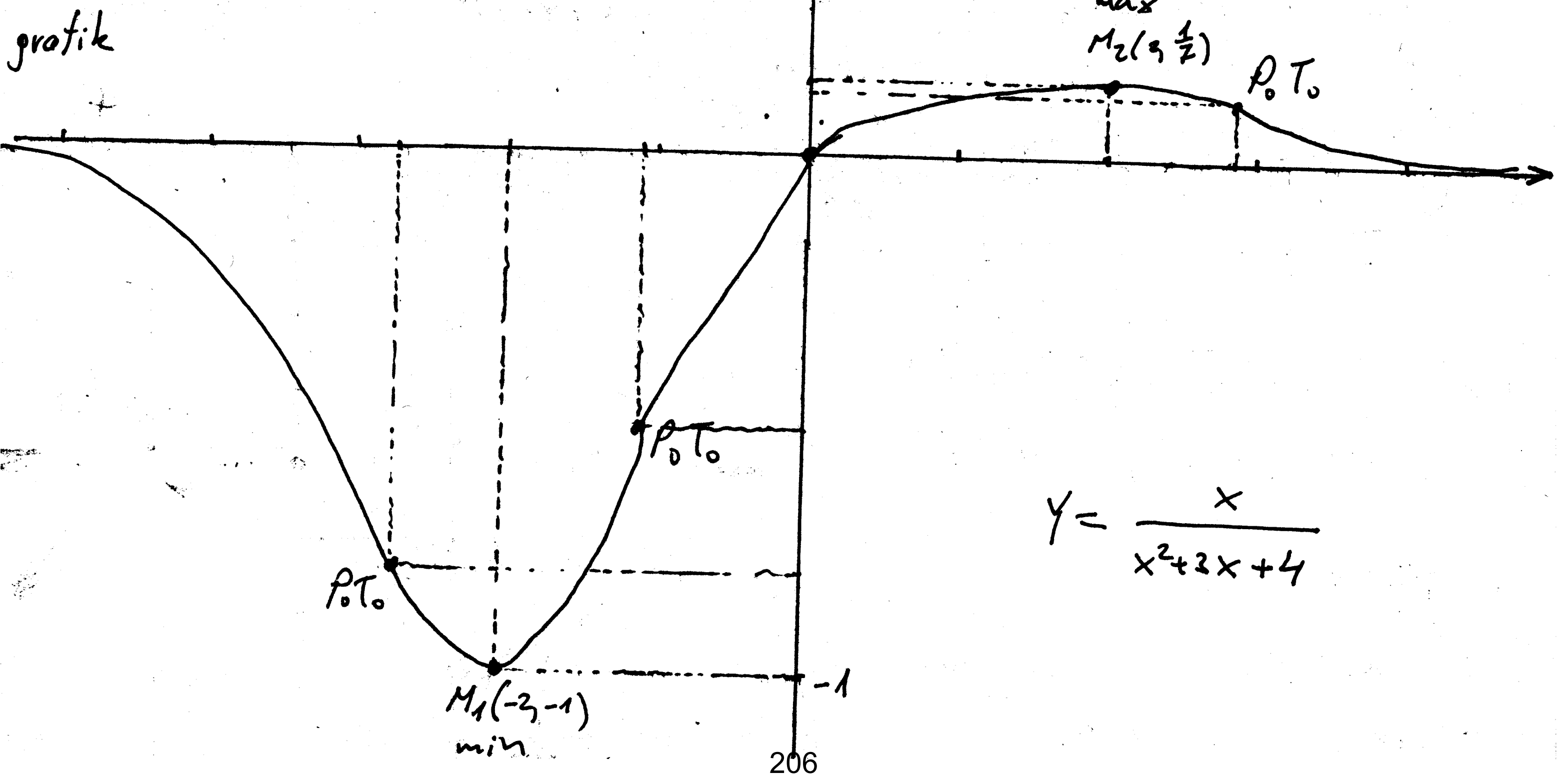
P_0T_0 P_0T_0 P_0T_0

$f(-2{,}77)\approx -0{,}82$

$f(3{,}88)\approx 0{,}13$

$f(-1{,}11)\approx -0{,}58$

grafik



(#) Ispitati i grafički predstaviti f-ju $y = xe^{\frac{1}{x}}$.

Rj. definiciono područje

$x \neq 0$, $\mathcal{D}: x \in \mathbb{R}\setminus\{0\}$

parnost, neparnost, periodičnost

$f(-x) = -xe^{\frac{1}{-x}} = -xe^{-\frac{1}{x}}$

f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa Y-osom, znak f-je

$xe^{\frac{1}{x}} = 0$

$x = 0$ ili $e^{\frac{1}{x}} = 0$

nije definisano $\quad e^{x} \neq 0 \ \forall x \in \mathbb{R}$

f-ja nema nulu

f(0) nije definisano

f-ja ne siječe Y-osu

$e^{\frac{1}{x}} > 0 \ \forall x \in \mathcal{D}$

X	$(-\infty,0)$	$(0,+\infty)$
Y	−	+

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

$x = 0$ f-ja ima prekid

$\left(-\frac{1}{x}\right)' = \left(-x^{-1}\right)'$

$$\lim_{x\to -0} f(x) = \lim_{x\to -0} xe^{\frac{1}{x}} = (-0)\cdot e^{\frac{1}{-0}} = (-0)\cdot e^{-\infty} = \frac{-0}{e^{\infty}} = \frac{-0}{\infty} = 0$$

$$\lim_{x\to +0} f(x) = \lim_{x\to +0} xe^{\frac{1}{x}} \left(=0\cdot\infty\right) = \lim_{x\to +0} \frac{x}{e^{-\frac{1}{x}}} \left(=\frac{0}{0}\right) \overset{L_oP_o}{=\!=} \lim_{x\to +0} \frac{1}{e^{-\frac{1}{x}}\cdot\frac{1}{x^2}} = \lim_{x\to +0} \frac{x^2}{e^{-\frac{1}{x}}} =$$

pokušat ćemo na drugi način:

$$\lim_{x\to +0} f(x) = \lim_{x\to +0} xe^{\frac{1}{x}} \left(=0\cdot\infty\right) = \lim_{x\to 0+} \frac{e^{\frac{1}{x}}}{x^{-1}} \left(=\frac{\infty}{\infty}\right) \overset{L_oP_o}{=\!=} \lim_{x\to 0+} \frac{e^{\frac{1}{x}}\cdot\left(\frac{1}{x}\right)'}{\left(\frac{1}{x}\right)'} = e^{\frac{1}{0+}} = \infty$$

$\Rightarrow x = 0$ je $V_o A_o$

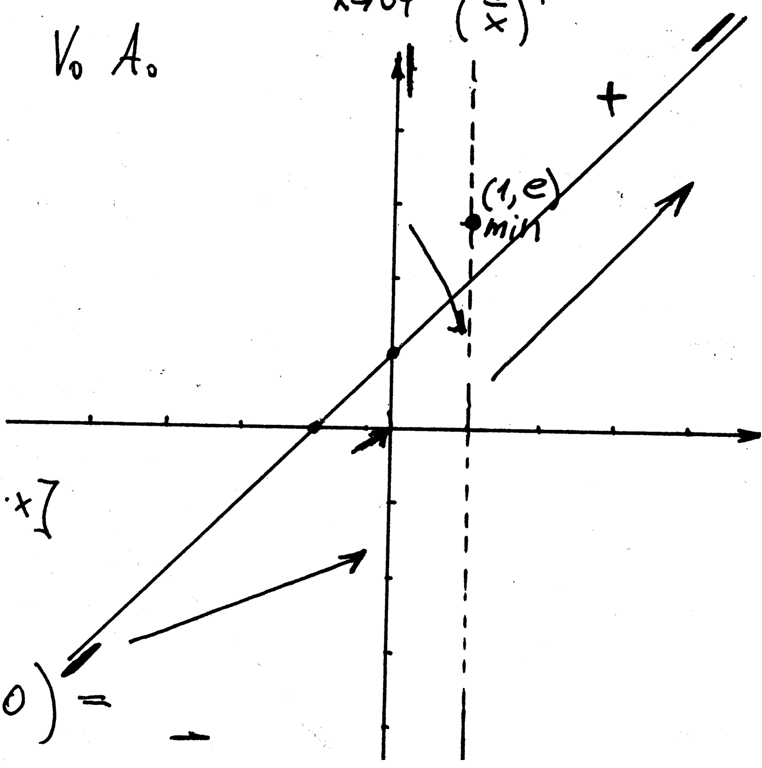
$$\left.\begin{array}{l} \lim\limits_{x\to -\infty} f(x) = \lim\limits_{x\to -\infty} xe^{\frac{1}{x}} = -\infty\cdot 1 = -\infty \\ \lim\limits_{x\to +\infty} f(x) = \lim\limits_{x\to +\infty} xe^{\frac{1}{x}} = +\infty\cdot 1 = \infty \end{array}\right\} \Rightarrow$$

$\Rightarrow$ f-ja nema $H_o A_o$

$y = kx + n$, $\quad k = \lim\limits_{x\to\infty} \frac{f(x)}{x}$, $\quad n = \lim\limits_{x\to\infty}\left[f(x) - k\cdot x\right]$

$$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} e^{\frac{1}{x}} = e^{0} = 1$$

$$n = \lim_{x\to\infty}\left[xe^{\frac{1}{x}} - x\right] = \lim_{x\to\infty} x\left(e^{\frac{1}{x}} - 1\right) \left(=\infty\cdot 0\right) =$$



$$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}}-1}{\frac{1}{x}} \left(=\frac{0}{0}\right) \overset{L.P.}{=} \lim_{x\to\infty} \frac{e^{\frac{1}{x}}\cdot\cancel{\left(\frac{1}{x}\right)'}}{\cancel{\left(\frac{1}{x}\right)'}} = \lim_{x\to\infty} e^{\frac{1}{x}} = e^{0} = 1$$

$Y = x + 1$ je K. A.

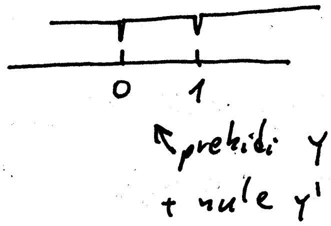
rast i opadanje

$$Y' = \left(x e^{\frac{1}{x}}\right)' = e^{\frac{1}{x}} + x\cdot e^{\frac{1}{x}}\cdot\left(x^{-1}\right)' = e^{\frac{1}{x}} + xe^{\frac{1}{x}}\cdot\left(-x^{-2}\right) = e^{\frac{1}{x}}\left(1+x\cdot\left(-\frac{1}{x^2}\right)\right)$$

$$Y' = e^{\frac{1}{x}}\left(1-\frac{1}{x}\right)$$

$Y' = 0$ akko $1-\frac{1}{x}=0$

$x = 1$



x	$(-\infty,0)$	$(0,1)$	$(1,+\infty)$
Y'	+	−	+
Y	↗	↘	↗
			MIN

rast i opadanje

ekstremi f-je

na osnovu tabele rasta i opadanja f-ja ima minimum u tački $(1, f(1))$. $f(1) = 1\cdot e^{\frac{1}{1}} = e$ $f_{min}(1) = e$ $(1, e)$

$e \approx 2,71$

prevojne tačke i intervali konveksnosti i konkavnosti

$$Y'' = \left(e^{\frac{1}{x}}\left(1-\frac{1}{x}\right)\right)' = e^{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)\left(1-\frac{1}{x}\right) + e^{\frac{1}{x}}\cdot\left(-\left(-\frac{1}{x^2}\right)\right)$$

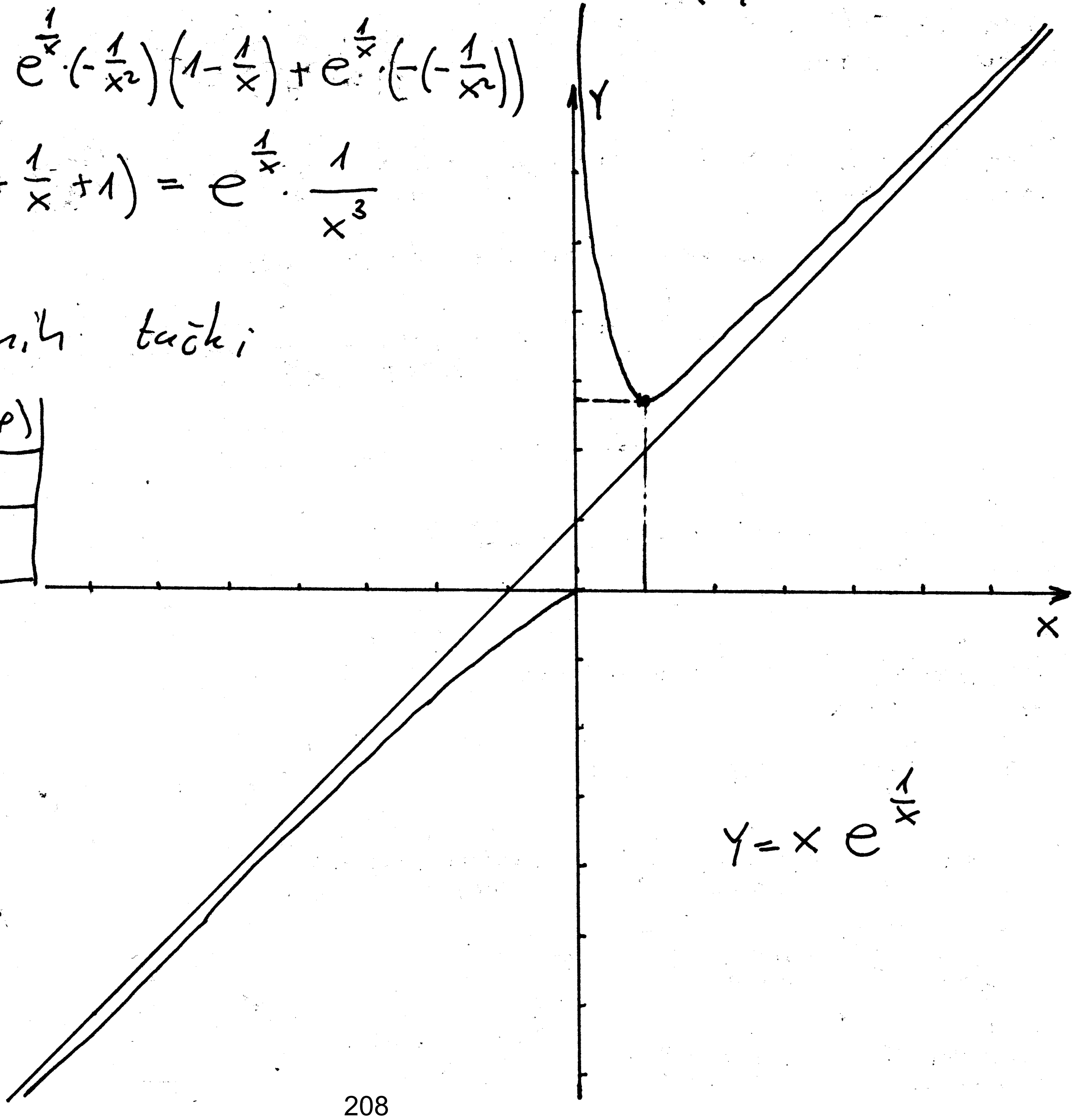
$$= e^{\frac{1}{x}}\cdot\frac{1}{x^2}\left(-1+\frac{1}{x}+1\right) = e^{\frac{1}{x}}\cdot\frac{1}{x^3}$$

$Y'' \neq 0 \quad \forall x \in D$

nema prevojnih tački

x	$(-\infty,0)$	$(0,+\infty)$
Y''	−	+
Y	∩	∪

grafik



(#) Ispitati f-ju i nacrtati joj grafik $y = x^3 e^{-\frac{x^2}{6}}$.

Rj. definiciono područje

$\mathcal{D}: x \in \mathbb{R}$

parnost, neparnost, periodičnost

$y(-x) = (-x)^3 e^{-\frac{(-x)^2}{6}} = -x^3 e^{-\frac{x^2}{6}}$

f-ja je neparna (simetrična u odnosu na koordinatni početak). Dovoljno ju je ispitati za $x > 0$. F-ja nije periodična

nule, presjek sa Y-osom, znak f-je

$x^3 \underbrace{e^{-\frac{x^2}{6}}}_{>0\ \forall x} = 0$

$x = 0$

(0,0) je nula f-je i presjek sa Y-osom

x	(−∞,0)	(0,+∞)
Y	−	+

znak f-je

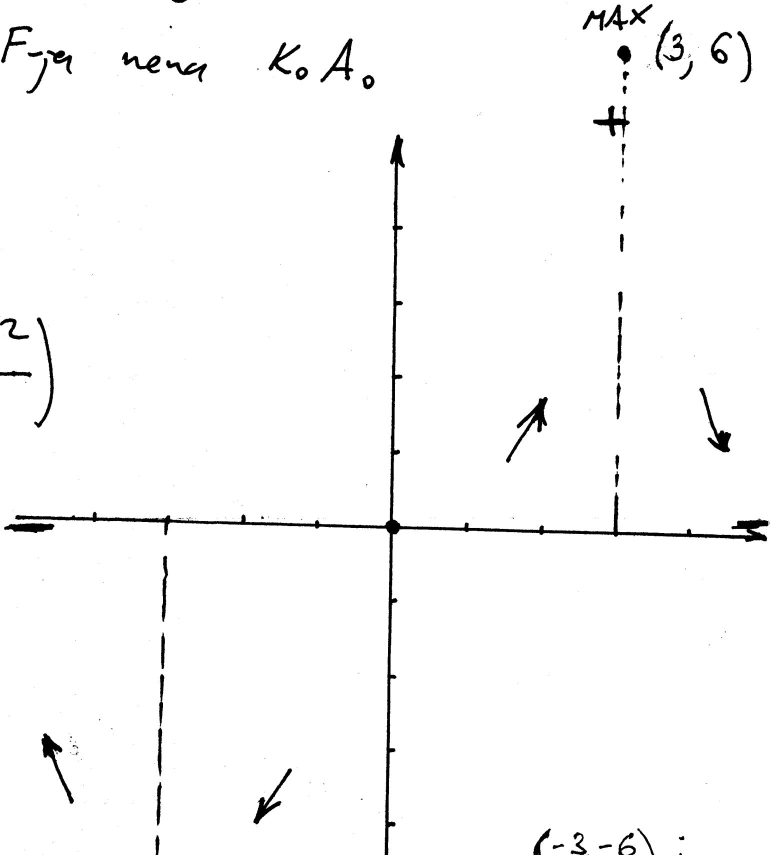
ponašanje na krajevima intervala definisanosti i asimptote

f-ja nema prekid ⇒ nema V. A.

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^3 e^{-\frac{x^2}{6}} = \lim_{x\to+\infty} \frac{x^3}{e^{\frac{x^2}{6}}} \left(\frac{+\infty}{\infty}\right) \overset{L.P.}{=} \lim_{x\to+\infty} \frac{3x^2}{e^{\frac{x^2}{6}}\cdot\frac{1}{6}2x} =$$

$$= \lim_{x\to+\infty} \frac{9x}{e^{\frac{x^2}{6}}} \left(\frac{\infty}{\infty}\right) \overset{L.P.}{=} \lim_{x\to\infty} \frac{9}{e^{\frac{x^2}{6}}\cdot\frac{1}{6}2x} = \lim_{x\to\infty} \frac{27}{xe^{\frac{x^2}{6}}} = 0$$

⇒ $x = 0$ je H. A., F-ja nema K. A.



rast i opadanje

$y' = 3x^2 e^{-\frac{x^2}{6}} + x^3 \cdot e^{-\frac{x^2}{6}} \cdot (-\frac{1}{6}) \cdot 2x$

$= 3x^2 e^{-\frac{x^2}{6}} - \frac{1}{3} x^4 e^{-\frac{x^2}{6}}$

$= x^2 e^{-\frac{x^2}{6}} \left(3 - \frac{1}{3}x^2\right) = x^2 e^{-\frac{x^2}{6}} \left(\frac{9 - x^2}{3}\right)$

$y' = 0 \Leftrightarrow x_1 = 0,\ x_2 = -3,\ x_3 = 3$

← prekidi y + nule y'

x	(0,3)	(3,+∞)
y'	+	−
Y	↗	↘
	MAX	

rast i opadanje

ekstremi f-je

Iz tabele rasta i opadanja vidimo da f-ja ima ekstrem za $x = 3$ $f(3) = 27e^{-\frac{9}{6}} = 27e^{-\frac{3}{2}} \approx 6$

(−3,−6) i (3,6) je maksimum f-je

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left(x^2 e^{-\frac{x^2}{6}} \frac{1}{3}(9-x^2)\right)' = 2x e^{-\frac{x^2}{6}} \cdot \frac{1}{3}(9-x^2) + x^2 e^{-\frac{x^2}{6}} \cdot \left(-\frac{1}{6}\right) 2x \cdot \frac{1}{3}(9-x^2) + x^2 e^{-\frac{x^2}{6}} \cdot \frac{1}{3}(-2x) =$$

$$= \frac{2}{3} x e^{-\frac{x^2}{6}}(9-x^2) - \frac{1}{9} x^3 e^{-\frac{x^2}{6}}(9-x^2) - \frac{2}{3} x^3 e^{-\frac{x^2}{6}} = x e^{-\frac{x^2}{6}}\left(\frac{2}{3}(9-x^2) - \frac{1}{9}x^2(9-x^2) - \right.$$

$$\left. \frac{2}{3}x^2\right) = x e^{-\frac{x^2}{6}} \cdot \frac{54-6x^2-9x^2+x^4-6x^2}{9} = x e^{-\frac{x^2}{6}} \cdot \frac{x^4-21x^2+54}{9}$$

$y''=0$ akko $x=0$ i $x^4-21x^2+54=0$

$x^2=t$

$t^2-21t+54=0$

$D=441-216=225$

$t_{1,2}=\frac{21\pm 15}{2}$

$t_1=\frac{36}{2}=18$ $t_2=\frac{6}{2}=3$

$x^2=18$

$x=\pm\sqrt{18}$

$x_1=3\sqrt{2}$ $x_2=-3\sqrt{2}$

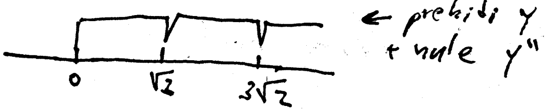
$x^2=3$

$x_3=-\sqrt{3}$

$x_4=\sqrt{3}\approx 1,73$

$3\sqrt{2}\approx 4,24$

f-ja simetrična u odnosu na koordinatni početak pa nas zanimaju samo pozitivne vrijednosti



x	(0, √2)	(√3, 3√2)	(3√2, +∞)
y''	+	−	+
y	∪	∩	∪

P_0T_0 P_0T_0 P_0T_0

$y=x^3 e^{-\frac{x^2}{6}}$

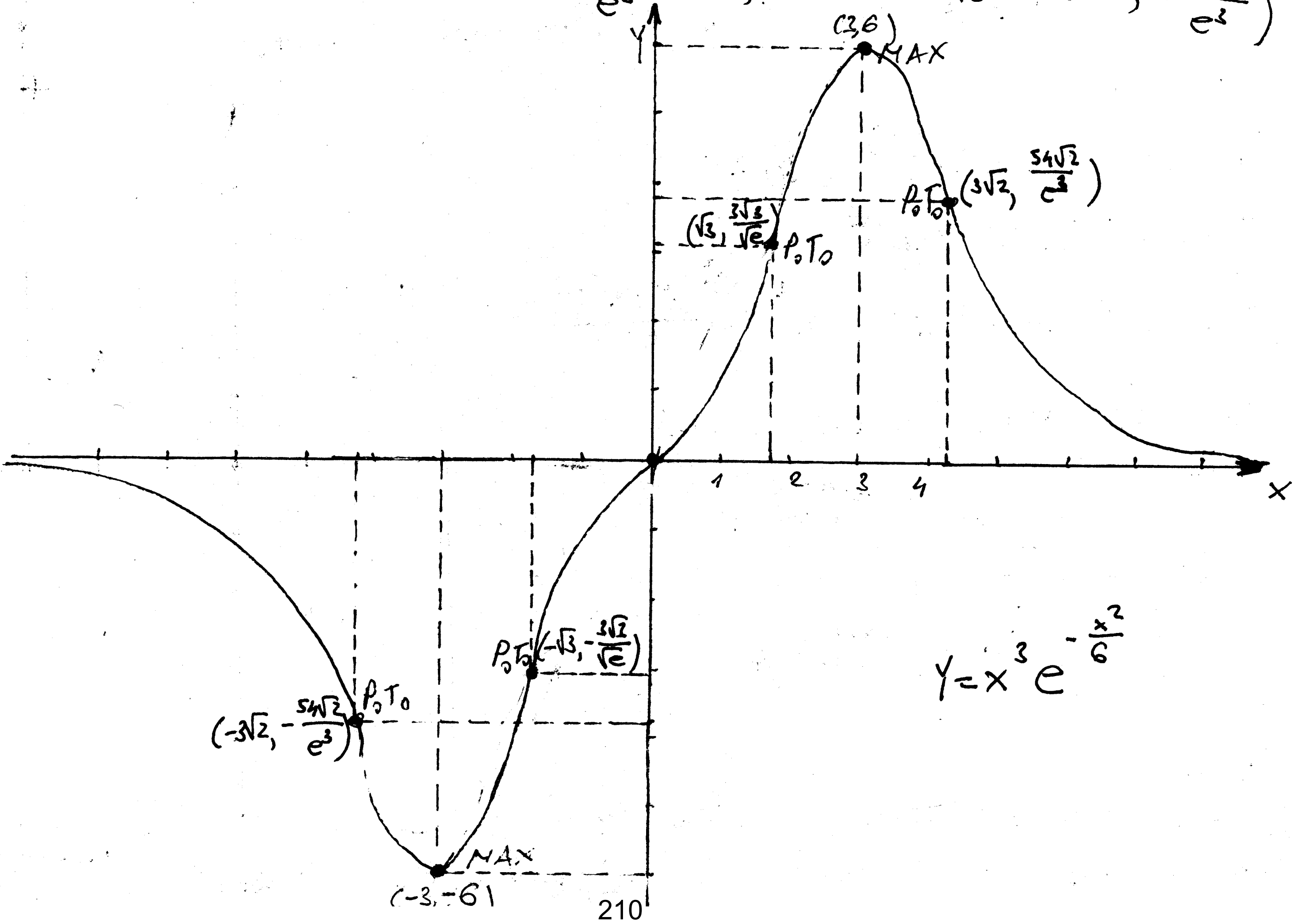
$y(0)=0$

$y(\sqrt{3})=3\sqrt{3}\, e^{-\frac{3}{6}}=\frac{3\sqrt{3}}{\sqrt{e}}\approx 3,15$

$y(3\sqrt{2})=27\cdot 2\sqrt{2}\; e^{-\frac{9\cdot 2}{6}}=54\sqrt{2}\, e^{-3}=\frac{54\sqrt{2}}{e^3}\approx 3,8$

Prevojne tačke su $(0,0)$, $\left(\sqrt{3}, \frac{3\sqrt{3}}{\sqrt{e}}\right)$, $\left(3\sqrt{2}, \frac{54\sqrt{2}}{e^3}\right)$, $\left(-\sqrt{3}, -\frac{3\sqrt{3}}{\sqrt{e}}\right)$ i $\left(-3\sqrt{2}, -\frac{54\sqrt{2}}{e^3}\right)$

grafik



(#) Ispitati i grafički predstaviti f-ju $y=\frac{1}{x}\ln x$.

Rj: definiciono područje

$x\neq 0,\quad x>0$

$\mathcal{D}: x\in(0,+\infty)$

parnost neparnost periodičnost

$\mathcal{D}$ nije simetrično ⇒

f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y=0$

$\frac{1}{x}\ln x=0$

$\ln x=0$

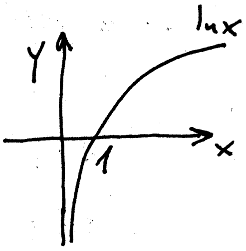
$x=e^0$

$x=1$

(1,0) je nula f-je

f(0) nije definisano

f-ja ne siječe y-osu



x	(0, 1)	(1, +∞)
ln x	−	+
y	−	+

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

$$\lim_{x\to+0} f(x)=\lim_{x\to+0}\frac{1}{x}\ln x\ \left(=\infty\cdot(-\infty)\right)=\frac{1}{+0}\ln(+0)=(+\infty)\cdot(-\infty)=-\infty$$

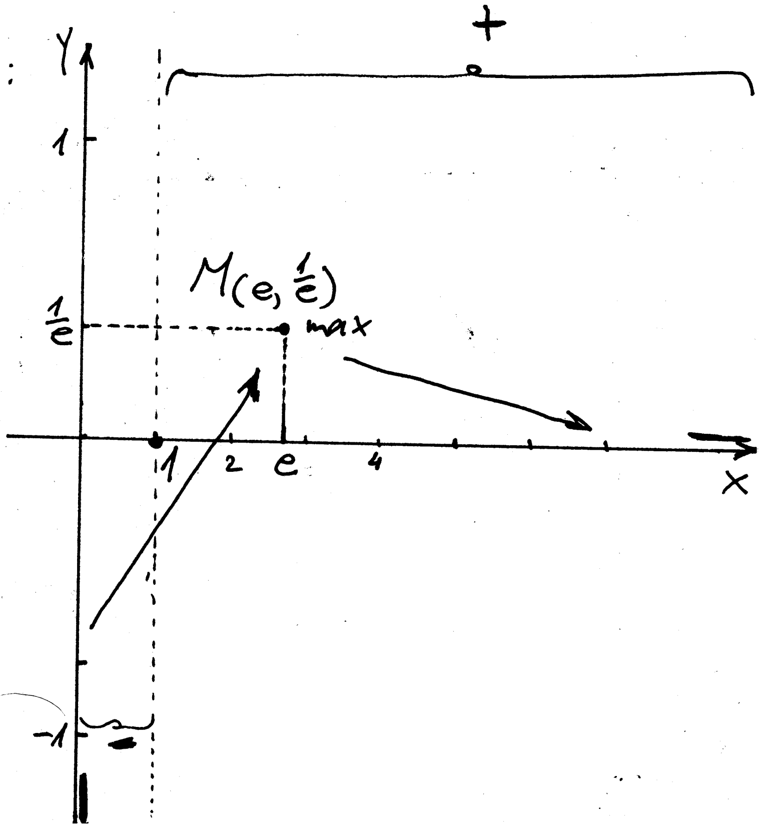
⇒ $x=0$ je V.A. (sa desne strane)

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{\ln x}{x}\ \left(=\frac{\infty}{\infty}\right)\overset{L.P.}{=}\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=0\ \Rightarrow$$

⇒ $y=0$ je H.A.

f-ja nema kosu asimptotu

počinjemo sa skiciranjem grafa:



rast i opadanje

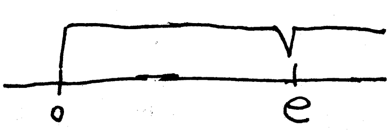
$$y'=\left(\frac{1}{x}\ln x\right)'=\left(\frac{\ln x}{x}\right)'=$$

$$=\frac{\frac{1}{x}\cdot x-\ln x}{x^2}=\frac{1-\ln x}{x^2}$$

$y'=0$ akko $1-\ln x=0$

$\ln x=1$

$x=e\approx 2{,}7183$



← nule y' + prekidi y

x	$(0, e)$	$(e, +\infty)$
y'	$+$	$-$
y	↗	↘
	max	

rast i opadanje

$$f(e) = \frac{1}{e}\ln e = \frac{1}{e} \approx 0,3679$$

ekstremi f-je

Na osnovu tabele rasta i opadanja, f-ja ima maksimum u tački $M(e, \frac{1}{e})$.

prevojne tačke i intervali konveksnosti i konkavnosti.

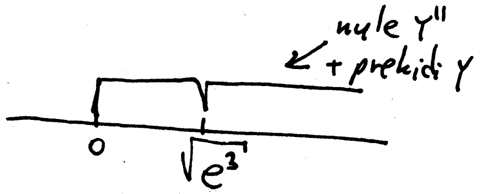
$$y'' = \left(\frac{1-\ln x}{x^2}\right)' = \frac{-\frac{1}{x}\cdot x^2 - (1-\ln x)2x}{x^4} = \frac{-\cancel{x} - (1-\ln x)\cdot 2\cancel{x}}{x^{\cancel{4}\,3}} = \frac{-1-2+2\ln x}{x^3}$$

$$y'' = \frac{2\ln x - 3}{x^3} \qquad y'' = 0 \text{ akko } 2\ln x - 3 = 0$$

$$2\ln x = 3$$

$$\ln x = \frac{3}{2}$$

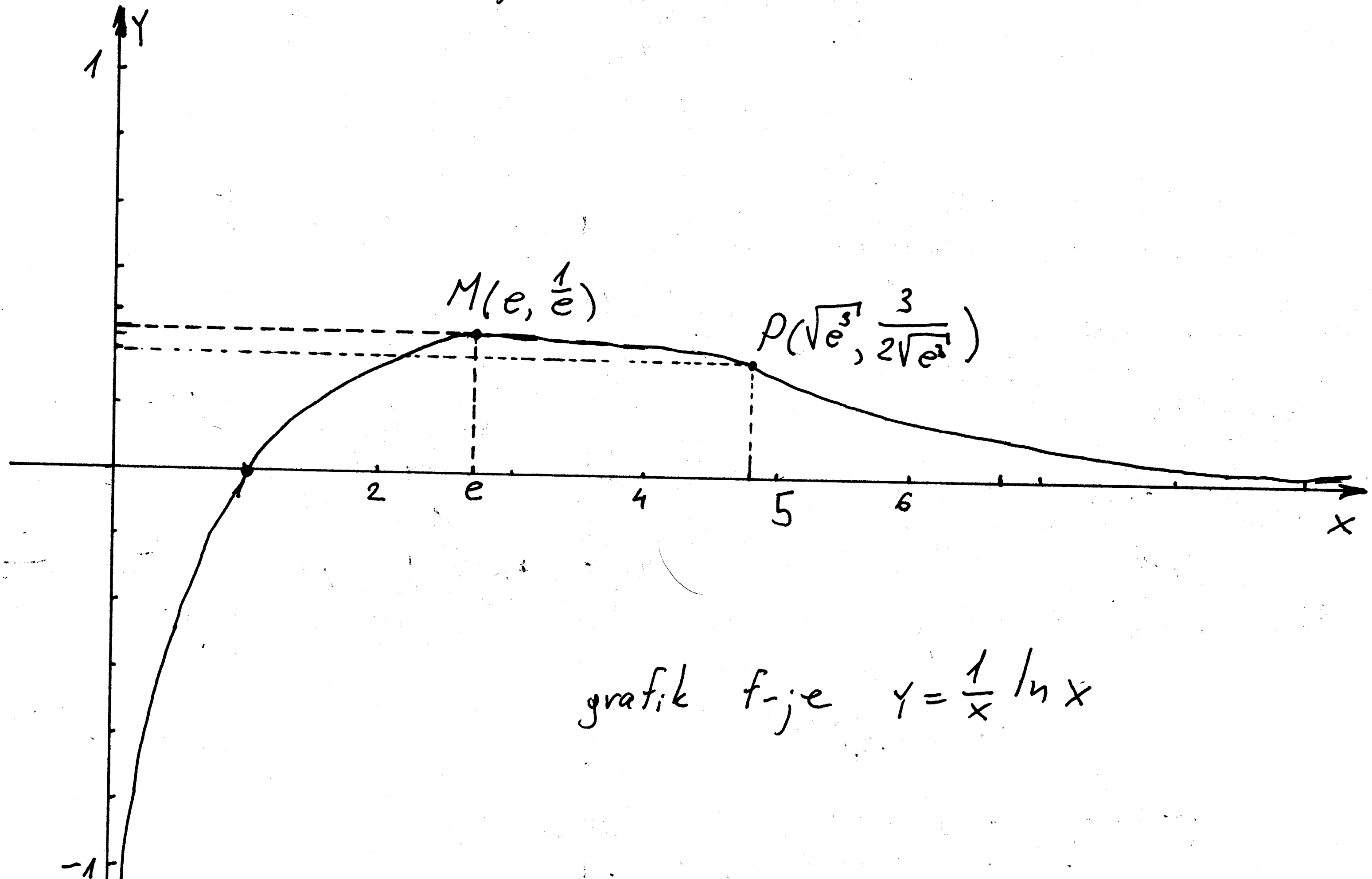
$$x = e^{\frac{3}{2}} = \sqrt{e^3} \approx 4,4817$$



x	$(0, \sqrt{e^3})$	$(\sqrt{e^3}, +\infty)$
y''	$-$	$+$
y	$\frown$	$\cup$
	P_0T_0	

$$f(e^{\frac{3}{2}}) = \frac{1}{\sqrt{e^3}}\cdot\frac{3}{2} = \frac{3}{2\sqrt{e^3}} \approx 0,3347.$$

$P(\sqrt{e^3}, \frac{3}{2\sqrt{e^3}})$ je prevojna tačka



grafik f-je $y = \frac{1}{x}\ln x$

(#) Ispitati f-ju i nacrtati joj grafik $y = \frac{\ln x - 1}{x^3}$.

Rj. definiciono područje

$x \neq 0 \quad x > 0$

$\mathcal{D}: x \in (0, +\infty)$

parnost, neparnost, periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$

$\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y = 0$ akko $\ln x - 1 = 0$

$\ln x = 1$

$x = e$

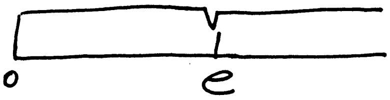
$(e, 0)$ nula f-je

$e \approx 2{,}7183$

$f(0) = ?$

$f(0)$ nije definisano

f-ja ne siječe y-osu



x	$(0, e)$	$(e, +\infty)$
$\ln x - 1$	$-$	$+$
x^3	$+$	$+$
y	$-$	$+$

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

$$\lim_{x \to +0} f(x) = \lim_{x \to +0} \frac{\ln x - 1}{x^3} \left(= \frac{-\infty - 1}{+0} \right) = \frac{-\infty}{+0} = -\infty \Rightarrow x = 0 \text{ je V. A.}$$

(sa desne strane)

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{\ln x - 1}{x^3} \left(= \frac{\infty}{\infty} \right) \overset{L.P.}{=\!=} \lim_{x \to +\infty} \frac{\frac{1}{x}}{3x^2} = \lim_{x \to +\infty} \frac{1}{3x^3} = \frac{1}{\infty} = 0$$

$\Rightarrow y = 0$ je H. A.

f-ja nema K. A.

počinjemo sa skiciranjem grafa

rast i opadanje

$$y' = \left(\frac{\ln x - 1}{x^3} \right)' = \frac{\frac{1}{x} \cdot x^{\overset{1}{\cancel{3}}} - (\ln x - 1) \cdot 3x^{\cancel{2}}}{x^{\cancel{6}4}}$$

$$y' = \frac{1 - 3\ln x + 3}{x^4} = \frac{4 - 3\ln x}{x^4}$$

$y' = 0$ akko $4 - 3\ln x = 0$

$3\ln x = 4$

$\ln x = \frac{4}{3}$ $\quad x = e^{\frac{4}{3}} = \sqrt[3]{e^4} \approx 3{,}7937$

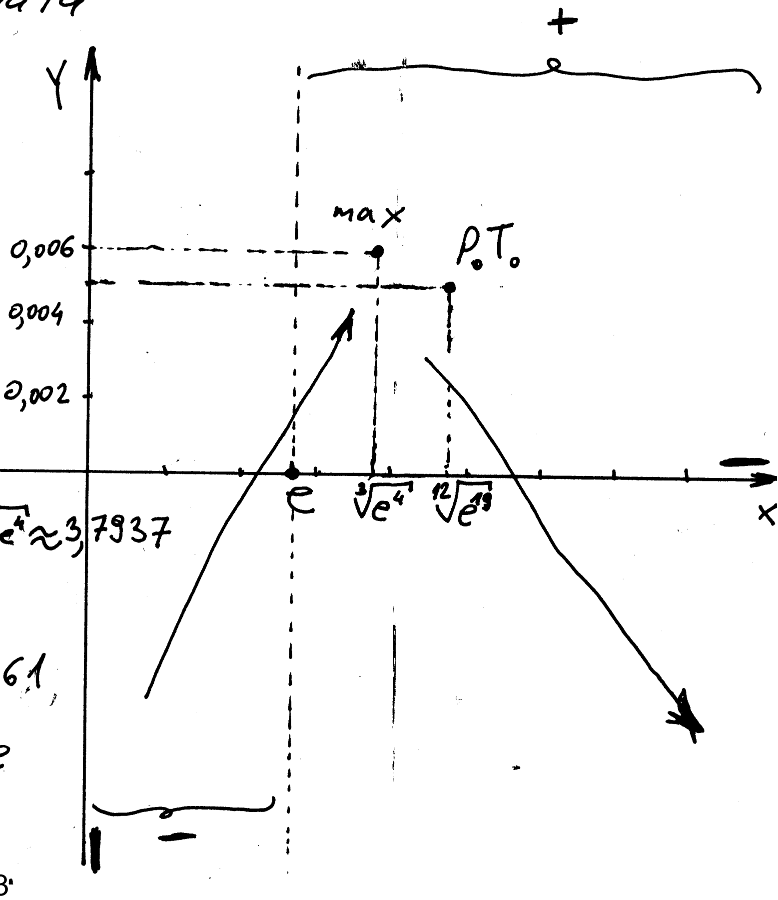
x	$(0, \sqrt[3]{e^4})$	$(\sqrt[3]{e^4}, +\infty)$
y'	$+$	$-$
y	$\nearrow$	$\searrow$

max

rast i opadanje

$\frac{1}{3e^4} \approx 0{,}0061$,

$$f(e^{\frac{4}{3}}) = \frac{\ln e^{\frac{4}{3}} - 1}{(\sqrt[3]{e^4})^3} = \frac{\frac{4}{3} - 1}{(\sqrt[3]{e^4})^3} = \frac{\frac{1}{3}}{e^4} = \frac{1}{3e^4}$$



ekstremi f-je

na osnovu tabele rasta i opadanja tačka $M\left(\sqrt[3]{e^4}, \frac{1}{3e^4}\right)$ je tačka maksimuma.

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left(\frac{4-3\ln x}{x^4}\right)' = \frac{-3\cdot\frac{1}{x}\cdot x^4-(4-3\ln x)\cdot 4x^3}{(x^4)^2} = \frac{-3\cancel{x^3}-(4-3\ln x)\cdot 4\cancel{x^3}}{x^5\cdot\cancel{x^3}} = \frac{-3-16+12\ln x}{x^5}$$

$$y'' = \frac{12\ln x - 19}{x^5}$$

$y''=0$ akko $12\ln x - 19 = 0$

$12\ln x = 19$

$\ln x = \frac{19}{12}$

$x = e^{\frac{19}{12}} = \sqrt[12]{e^{19}} \approx 4,8712$

x	$(0, \sqrt[12]{e^{19}})$	$(\sqrt[12]{e^{19}}, +\infty)$
y''	$-$	$+$
y	$\frown$	$\smile$

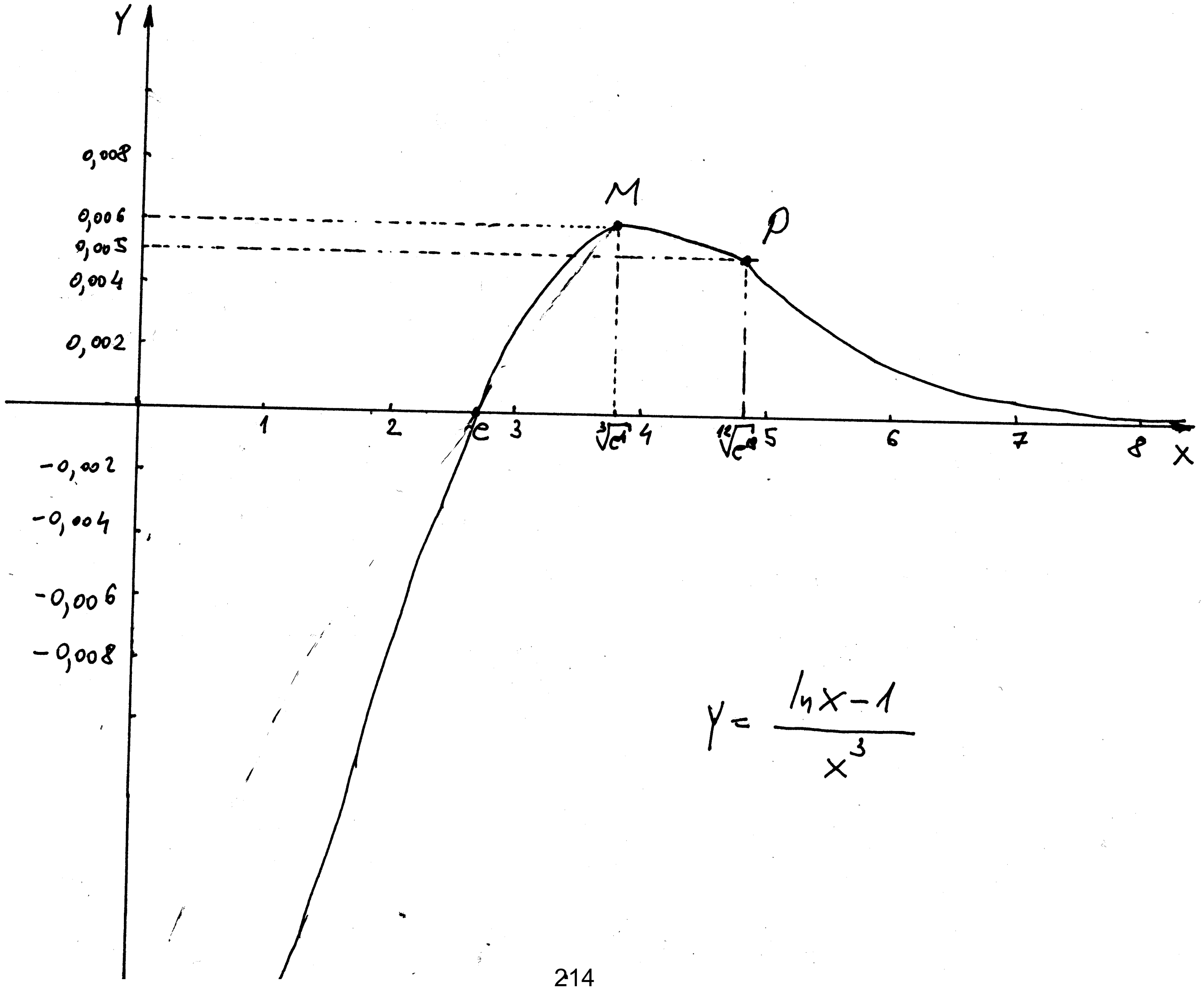
P.T.

intervali konveksnosti i konkavnosti

$$f(e^{\frac{19}{12}}) = \frac{\ln e^{\frac{19}{12}}-1}{(e^{\frac{19}{12}})^3} = \frac{\frac{19}{12}-1}{e^{\frac{19}{4}}} = \frac{\frac{7}{12}}{e^{\frac{19}{4}}} = \frac{7}{12\sqrt[4]{e^{19}}} \approx 0,005$$

$P\left(\sqrt[12]{e^{19}}, \frac{7}{12\sqrt[4]{e^{19}}}\right)$ je prevojna tačka

grafik



(#) Ispitati f-ju i nacrtati joj grafik $Y=\frac{x}{x-1}\ln\frac{x}{x-1}$
(bez analize drugog izvoda).

Rj: definiciono područje

parnost, neparnost, periodičnost

$x-1\neq 0$
$x\neq 1$ $\qquad \frac{x}{x-1}>0$ $\qquad \mathcal{D}: x\in(-\infty,0)\cup(1,+\infty)$

$\mathcal{D}$ nije simetrično $\Rightarrow$
$\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

$x>0$ $\quad x<0$
$x>1$ $\quad x<1$

0 1 $\quad$ 0 1 $\quad$ 0 1

$\ln\frac{x}{x-1}>0$ $\qquad \frac{x}{x-1}-1>0$

$\ln\frac{x}{x-1}>\ln 1$ $\qquad \frac{x-x+1}{x-1}>0$

$\frac{x}{x-1}>1$ $\qquad \frac{1}{x-1}>0$

$x-1>0$

$x>1$

nule, presjek sa Y-osom, znak f-je

$Y=0$ akko $x=0$
za $x=0$ f-ja nije definisana
f-ja nema nulu i ne sječe Y-osu

ponašanje na krajevima intervala definisanosti i asimptote

x	$(-\infty,0)$	$(1,+\infty)$
x	$-$	$+$
$x-1$	$-$	$+$
$\ln\frac{x}{x-1}$	$-$	$+$
Y	$-$	$+$

znak f-je

$$\lim_{x\to 0-}f(x)=\lim_{x\to 0-}\frac{x}{x-1}\ln\frac{x}{x-1}\left(=(0-)\cdot(-\infty)\right)$$

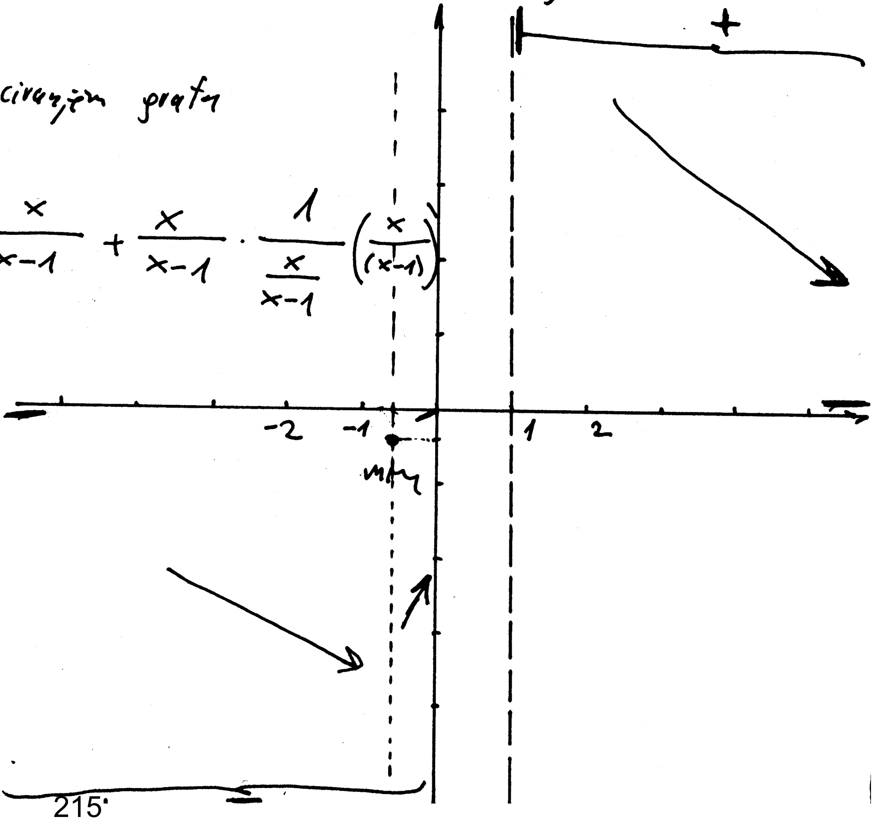
$$=\lim_{x\to 0-}\frac{\ln\frac{x}{x-1}}{\frac{x-1}{x}}\left(=\frac{-\infty}{\infty}\right)\overset{LoPo}{=}\lim_{x\to 0-}\frac{\frac{1}{\frac{x}{x-1}}\left(\frac{x}{x-1}\right)'}{\left(\frac{x-x+1}{x^2}\right)}=\lim_{x\to 0-}\frac{\frac{\cancel{x-1}}{x}\cdot\frac{\cancel{x}-1-\cancel{x}}{(x-1)^{\cancel{2}}}}{\frac{1}{x^2}}=\lim_{x\to 0-}\frac{(-1)x^{\cancel{2}}}{\cancel{x}(x-1)}=0$$ nema V.A. za $x=0$

$$\lim_{x\to 1+0}f(x)=\lim_{x\to 1+0}\frac{x}{x-1}\ln\frac{x}{x-1}=(+\infty)\cdot(+\infty)=+\infty \Rightarrow x=1 \text{ je V.A.}$$

$$\lim_{x\to+\infty}f(x)=\lim_{x\to+\infty}\frac{x}{x-1}\ln\frac{x}{x-1}=\lim_{x\to+\infty}\frac{1}{1-\frac{1}{x}}\ln\frac{1}{1-\frac{1}{x}}=1\cdot\ln 1=1\cdot 0=0 \Rightarrow Y=0 \text{ je H.A.}$$

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{1}{1-\frac{1}{x}}\ln\frac{1}{1-\frac{1}{x}}=1\cdot\ln 1=1\cdot 0=0 \Rightarrow Y=0 \text{ je H.A.}$$

f-ja nema kose asimptote
nakon ovog koraka počinjemo sa skiciranjem grafa



rast i opadanje

$$Y'=\left(\frac{x}{x-1}\ln\frac{x}{x-1}\right)'=\frac{x-1-x}{(x-1)^2}\ln\frac{x}{x-1}+\frac{x}{x-1}\cdot\frac{1}{\frac{x}{x-1}}\left(\frac{x}{(x-1)}\right)'$$

$$Y'=\frac{-1}{(x-1)^2}\ln\frac{x}{x-1}+\frac{-1}{(x-1)^2}$$

$$Y'=\frac{-1}{(x-1)^2}\left(\ln\frac{x}{x-1}+1\right)$$

$Y'=0$ akko $\ln\frac{x}{x-1}+1=0$

$\ln\frac{x}{x-1}=-1$

$\frac{x}{x-1}=e^{-1}$

$$\frac{x}{x-1}-\frac{1}{e}=0$$

$$\frac{ex-(x-1)}{e(x-1)}=0$$

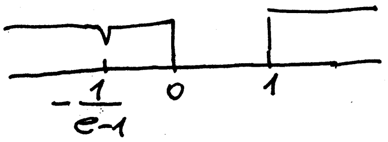
$$ex-x+1=0$$

$$x(e-1)=-1$$

$$x=-\frac{1}{e-1}\approx-0,5820$$

x	$(-\infty,-\frac{1}{e-1})$	$(-\frac{1}{e-1},0)$	$(1,+\infty)$
y'	$-$	$+$	$-$
y	↘	↗	↘

rast i opadanje



← prekidi y
+ nule y'

$$\ln\frac{-\frac{1}{e^{-1}-1}}{-\frac{1}{e^{-1}-1}-1}=e \qquad \ln\frac{5}{4}\approx 0{,}22$$

$$e>e^{-1}$$

$$e-1>e^{-1}-1$$

$$\frac{1}{e-1}<\frac{1}{e^{-1}-1} \quad /\cdot(-1)$$

$$-\frac{1}{e-1}>-\frac{1}{e^{-1}-1}$$

$$\ln\frac{-\frac{1}{e-1}}{-\frac{1}{e-1}-1}=e^{-1}$$

$$f\left(-\frac{1}{e-1}\right)=\frac{-\frac{1}{e-1}}{-\frac{1}{e-1}-1}\ln\frac{\frac{-1}{e-1}}{\frac{-1-(e-1)}{e-1}}=\frac{\frac{-1}{e-1}}{\frac{-e}{e-1}}\ln\frac{1}{e}=\frac{1}{e}\cdot(-1)=-\frac{1}{e}\approx-0,3679$$

ekstremi f-je

Na osnovu tabele rasta i opadanja tačka minimuma je $(-\frac{1}{e-1},-\frac{1}{e})$

prevojne tačke i intervali konveksnosti i konkavnosti

$$y''=\left[-(x-1)^{-2}\left(\ln\frac{x}{x-1}+1\right)\right]'=2(x-1)^{-3}\left(\ln\frac{x}{x-1}+1\right)+\left(-(x-1)^{-2}\right)\frac{x-1}{x}\cdot\frac{-1}{(x-1)^2}$$

$$y''=2(x-1)^{-3}\left(\ln\frac{x}{x-1}+1\right)-(x-1)^{-1}\cdot\frac{-1}{x(x-1)^2}=\frac{1}{(x-1)^3}\left[2\left(\ln\frac{x}{x-1}+1\right)+\frac{1}{x}\right]$$

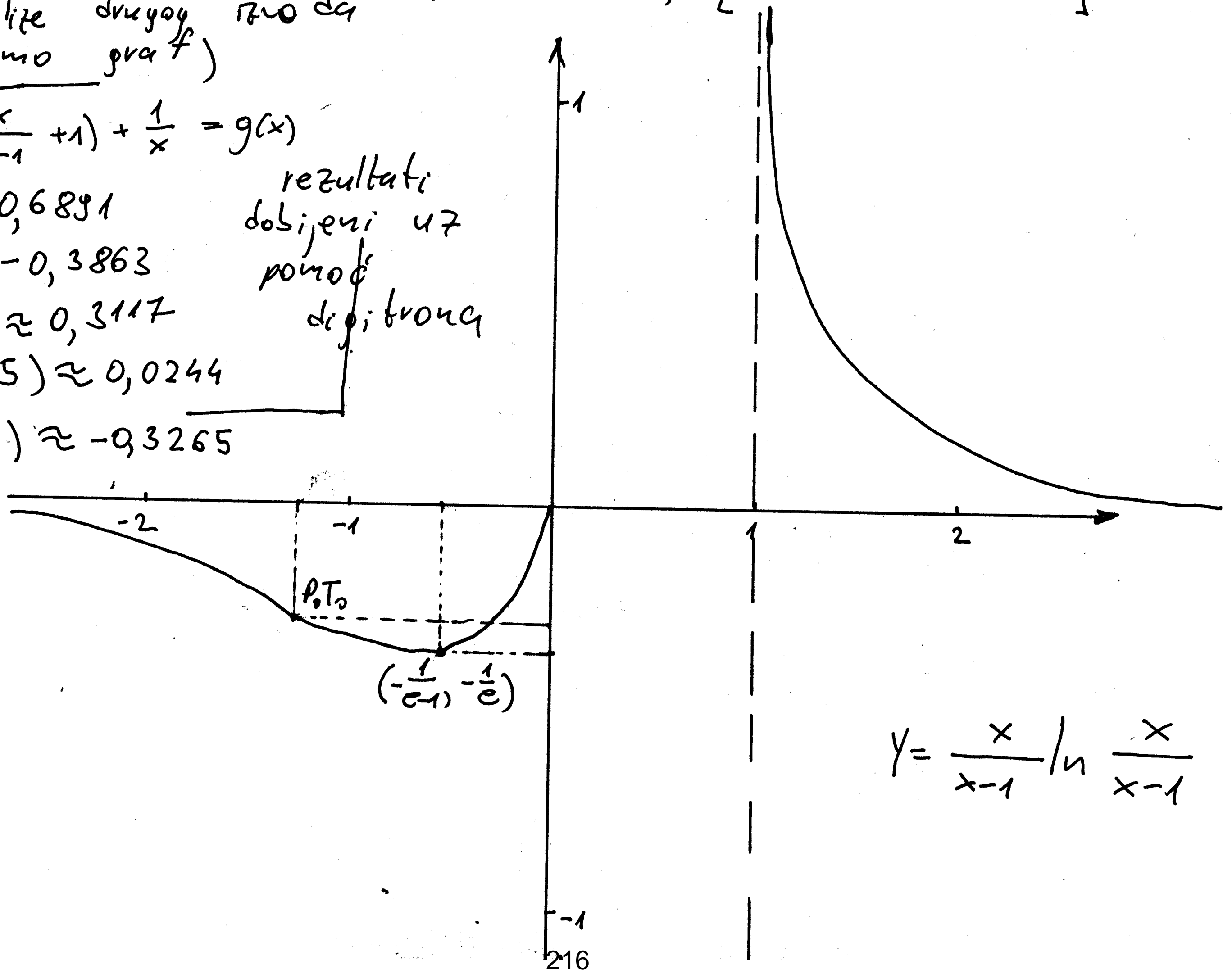
bez analize drugog izvoda
(crtamo graf)

$$2\left(\ln\frac{x}{x-1}+1\right)+\frac{1}{x}=g(x)$$

$g(-2)\approx0,6891$
$g(-1)\approx-0,3863$
$g(-1,5)\approx0,3117$
$g(-1,25)\approx0,0244$

rezultati dobijeni uz pomoć digitrona

$f(-1,25)\approx-0,3265$



(#) Ispitati f-ju i nacrtati njen grafik $Y = \frac{x^2+10}{x^2+4x+4}$.

Rj. $Y = \frac{x^2+10}{x^2+4x+4} = \frac{x^2+10}{(x+2)^2}$

definiciono područje

$x+2 \neq 0$ $\quad \mathcal{D}: x \in (-\infty,-2) \cup (-2,+\infty)$

$x \neq -2$

·parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa Y-osom i znak f-je

$Y=0 \Rightarrow x^2+10=0$

Kako je $x^2+10>0 \ \forall x \in \mathcal{D}$ to f-ja nema nule

$f(0) = \frac{0+10}{(0+2)^2} = \frac{10}{4} = \frac{5}{2}$

$(0, \frac{5}{2})$ je presjek sa Y-osom

$x^2+10>0 \ \forall x \in \mathcal{D}$ $\quad$ f-ja je

$(x+2)^2>0 \ \forall x \in \mathcal{D}$ $\quad$ uvijek pozitivna

ponašanje na krajevima intervala definisanosti i asimptote

za $x=-2$ f-ja ima prekid

$\lim_{x\to-2-0} f(x) = \lim_{x\to-2-0} \frac{x^2+10}{(x+2)^2} = \frac{(-2-0)^2+10}{(-2-0+2)^2} = \frac{14+0}{+0} = +\infty \Rightarrow x=-2$ je V. A. (sa lijeve strane)

$\lim_{x\to-2+0} f(x) = \lim_{x\to-2+0} \frac{x^2+10}{(x+2)^2} = \frac{(-2+0)^2+10}{(-2+0+2)^2} = \frac{14-0}{+0} = +\infty \Rightarrow x=-2$ je V. A. (sa desne strane)

$\lim_{x\to\mp\infty} f(x) = \lim_{x\to\mp\infty} \frac{x^2+10 :x^2}{x^2+4x+4 :x^2} = \lim_{x\to\mp\infty} \frac{1+\overset{0}{\frac{10}{x^2}}}{1+\underset{0}{\frac{4}{x}}+\underset{0}{\frac{4}{x^2}}} = 1 \Rightarrow Y=1$ je H. A.

f-ja nema kosu asimptotu

Poslije ovog koraka počinjemo skicirati grafik.

rast i opadanje

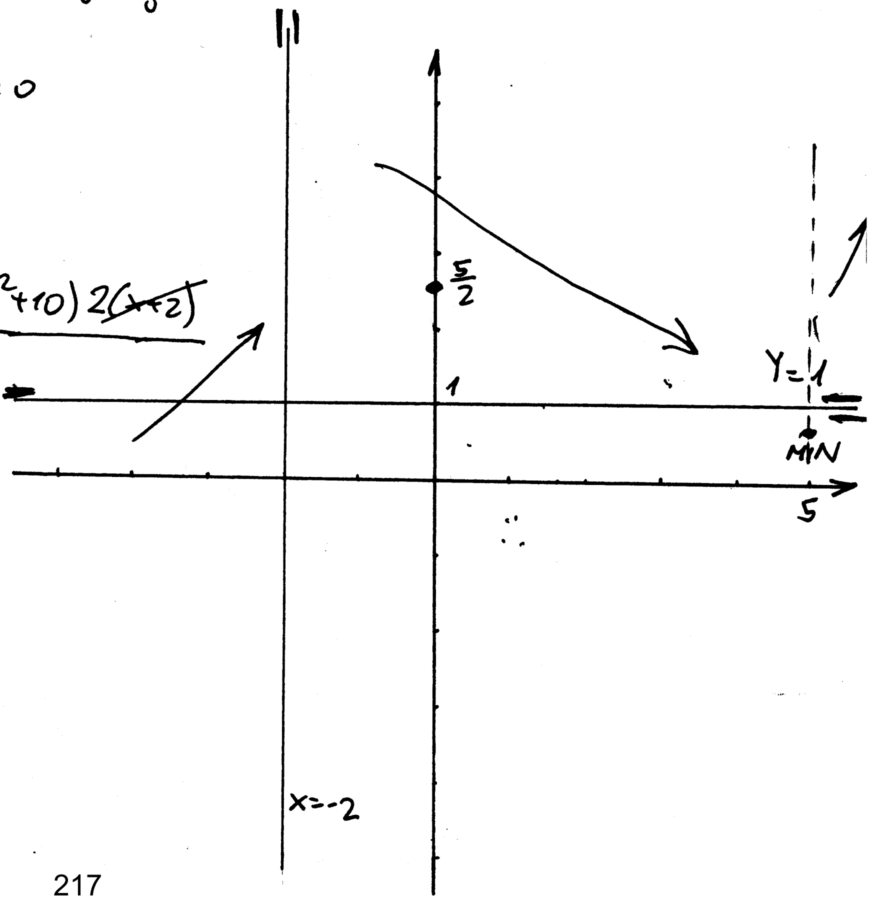
$Y' = \left(\frac{x^2+10}{(x+2)^2}\right)' = \frac{2x\cdot(x+2)^{\cancel{2}} - (x^2+10)2\cancel{(x+2)}}{(x+2)^{\cancel{4}3}}$

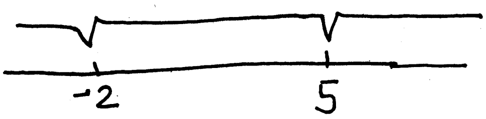
$Y' = \frac{\cancel{2x^2}+4x-\cancel{2x^2}-20}{(x+2)^3}$

$Y' = \frac{4x-20}{(x+2)^3} = 4\frac{x-5}{(x+2)^3}$

$Y'=0$ akko $x-5=0$

$x=5$





← prekidi Y + nule Y'

x	$(-\infty,-2)$	$(-2,5)$	$(5,+\infty)$
Y'	+	—	+
Y	↗	↘	↗

rast i opadanje

min

ekstremi f-je

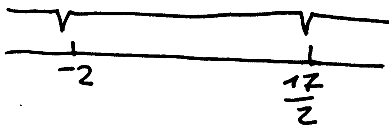
Stacionarna tačka je $x=5$.
Na osnovu tabele rasta i opadanja vidimo da f-ja u toj tački ima ekstrem i to minimum

$$f(5)=\frac{25+10}{7^2}=\frac{35}{49}\approx 0,71 \qquad \left(5,\frac{35}{49}\right) \text{ je tačka minimuma}$$

prevojne tačke i intervali konveksnosti i konkavnosti

$$Y''=\left(4\,\frac{x-5}{(x+2)^3}\right)'=4\,\frac{1\cdot(x+2)^{\cancel{3}}-(x-5)3\cancel{(x+2)^2}}{(x+2)^{\cancel{6}4}}=4\,\frac{x+2-3x+15}{(x+2)^4}.$$

$$Y''=4\,\frac{-2x+17}{(x+2)^4}=-4\,\frac{2x-17}{(x+2)^4}$$



← nule Y'' + prekidi Y

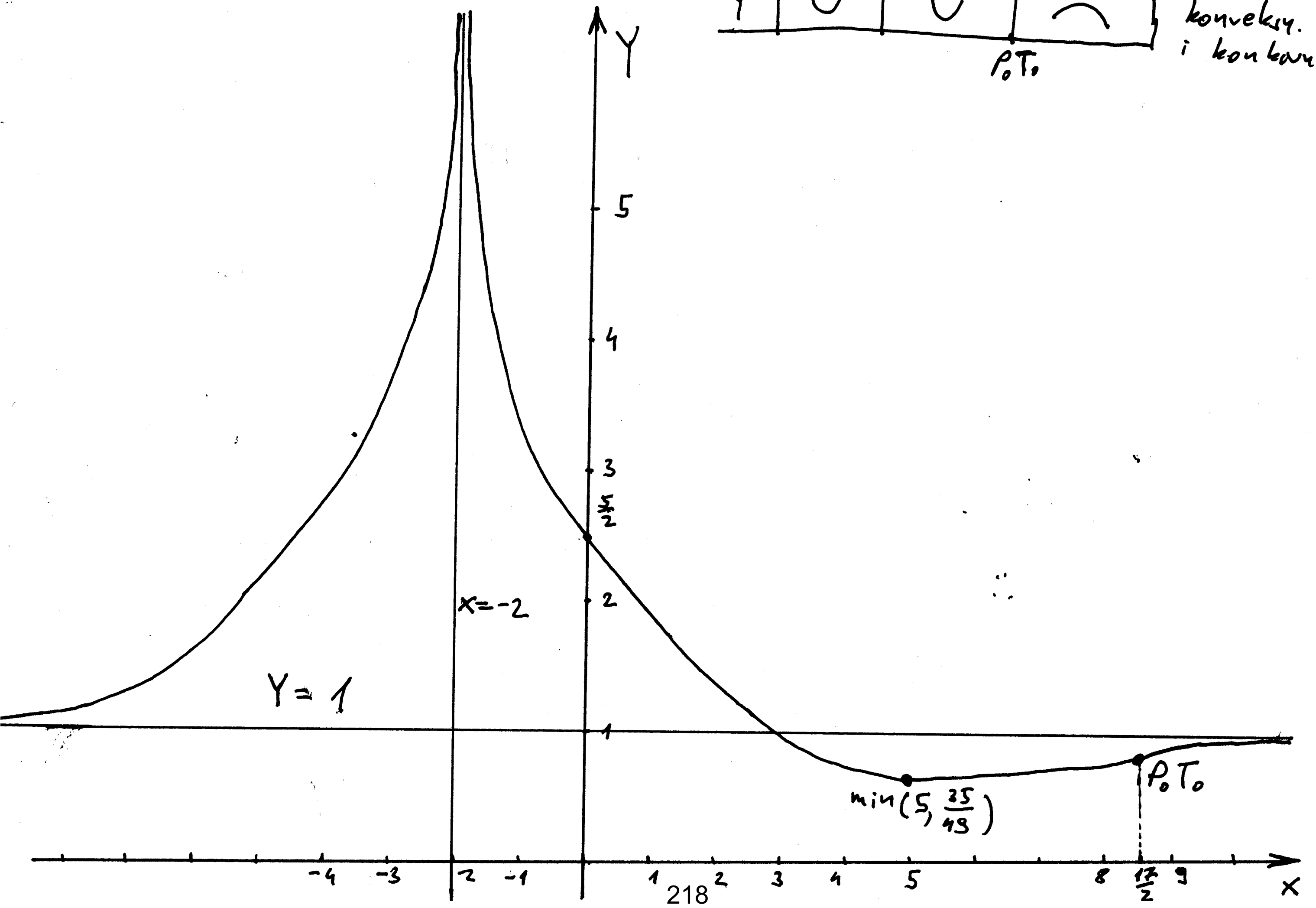
$Y''=0$ akko $2x-17=0$

$$x=\frac{17}{2}$$

x	$(-\infty,-2)$	$(-2,\frac{17}{2})$	$(\frac{17}{2},+\infty)$
Y''	+	+	—
Y	∪	∪	∩

intervali konveksn. i konkavn.

PoTo



(#) Ispitati f-ju i nacrtati njen grafik: $y = \frac{x^3-2}{2x^2}$.

Rj. definiciono područje

$D: x \neq 0$

parnost (neparnost), periodičnost

$f(-x) = \frac{(-x)^3-2}{2(-x)^2} = \frac{-x^3-2}{2x^2} \neq \pm f(x)$

• f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak

$y=0$ akko $x^3-2=0$

$x = \sqrt[3]{2} \approx 1,26$

$(\sqrt[3]{2}, 0)$ je nula f-je

$f(0) =$ nije definisano

f-ja ne siječe y-osu

$2x^2 > 0 \quad \forall x \in D$

$$\left.\begin{array}{ll} y>0 & \text{za} \quad x>\sqrt[3]{2} \\ y<0 & \text{za} \quad x<\sqrt[3]{2} \end{array}\right\} \text{znak f-je.}$$

ponašanje na krajevima intervala definisanosti i asimptote

za $x=0$ f-ja ima prekid

$$\left.\begin{array}{l} \lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{x^3-2}{2x^2} = \frac{(0-)^3-2}{2(0-)^2} = \frac{-2-0}{0+} = -\infty \\ \lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{(0+)^3-2}{2(0+)^2} = \frac{-2+0}{+0} = -\infty \end{array}\right\} \Rightarrow x=0 \text{ je V.A.}$$

$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^3-2}{2x^2} \frac{/:x^2}{/:x^2} = \pm\infty$ f-ja nema H.A.

Tražimo kosu asimptotu u obliku $y = kx+n$.

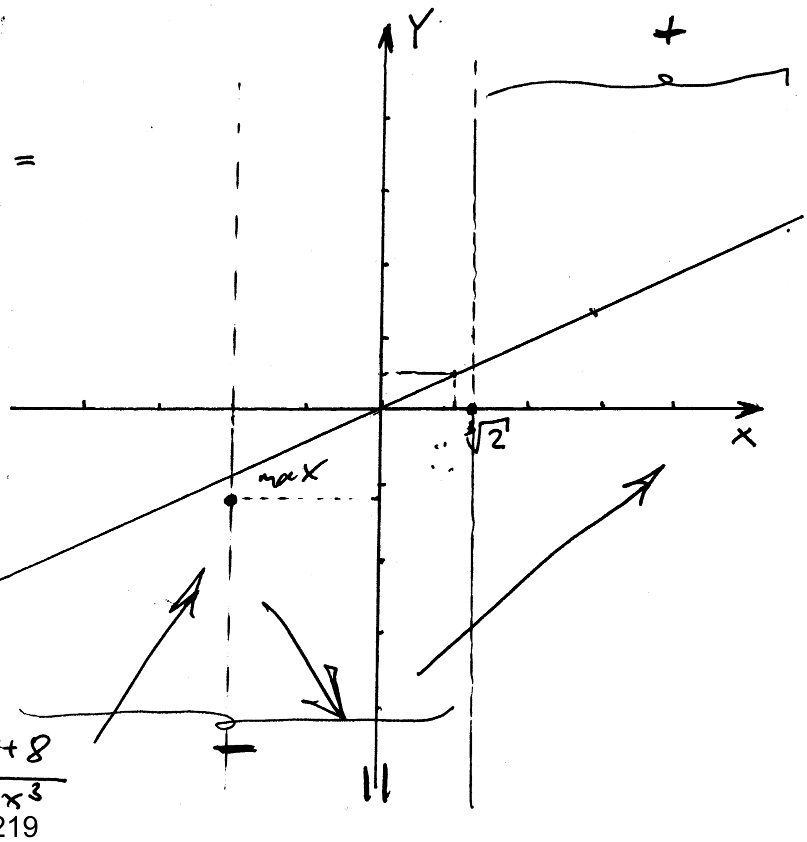
$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^3-2}{2x^3} \frac{/:x^3}{/:x^3} = \frac{1}{2}$

$n = \lim_{x\to\infty} [f(x) - kx] = \lim_{x\to\infty} \left[\frac{x^3-2}{2x^2} - \frac{1}{2}x\right] =$

$= \lim_{x\to\infty} \frac{x^3-2-x^3}{2x^2} = \lim_{x\to\infty} \frac{-2}{2x^2} = 0$

kosa asimptota je $y = \frac{1}{2}x$

Poslije ovog koraka počinjemo skicirati grafik.



rast i opadanje

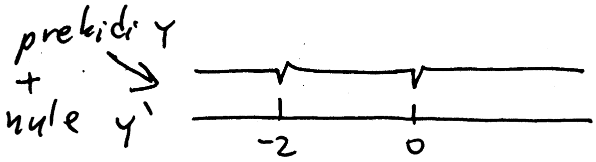
$y' = \left(\frac{x^3-2}{2x^2}\right)' = \frac{3x^2 \cdot 2x^2 - (x^3-2)4x}{4x^4} =$

$= \frac{6x^4-4x^4+8x}{4x^4} = \frac{2x^4+8x}{4x^4} = \frac{x^3+8}{2x^3}$

$y' = \frac{x^3+8}{2x^3}$, $y'=0$ akko $x^3+8=0$

$x^3 = -8$

$x = -2$



x	$(-\infty,-2)$	$(-2,0)$	$(0,+\infty)$
y'	+	−	+
y	↗	↘	↗
	max	N.D.	

$f(-2) = \frac{(-2)^3-2}{2(-2)^2} = \frac{-10}{8} = -\frac{5}{4} \approx -1{,}25$

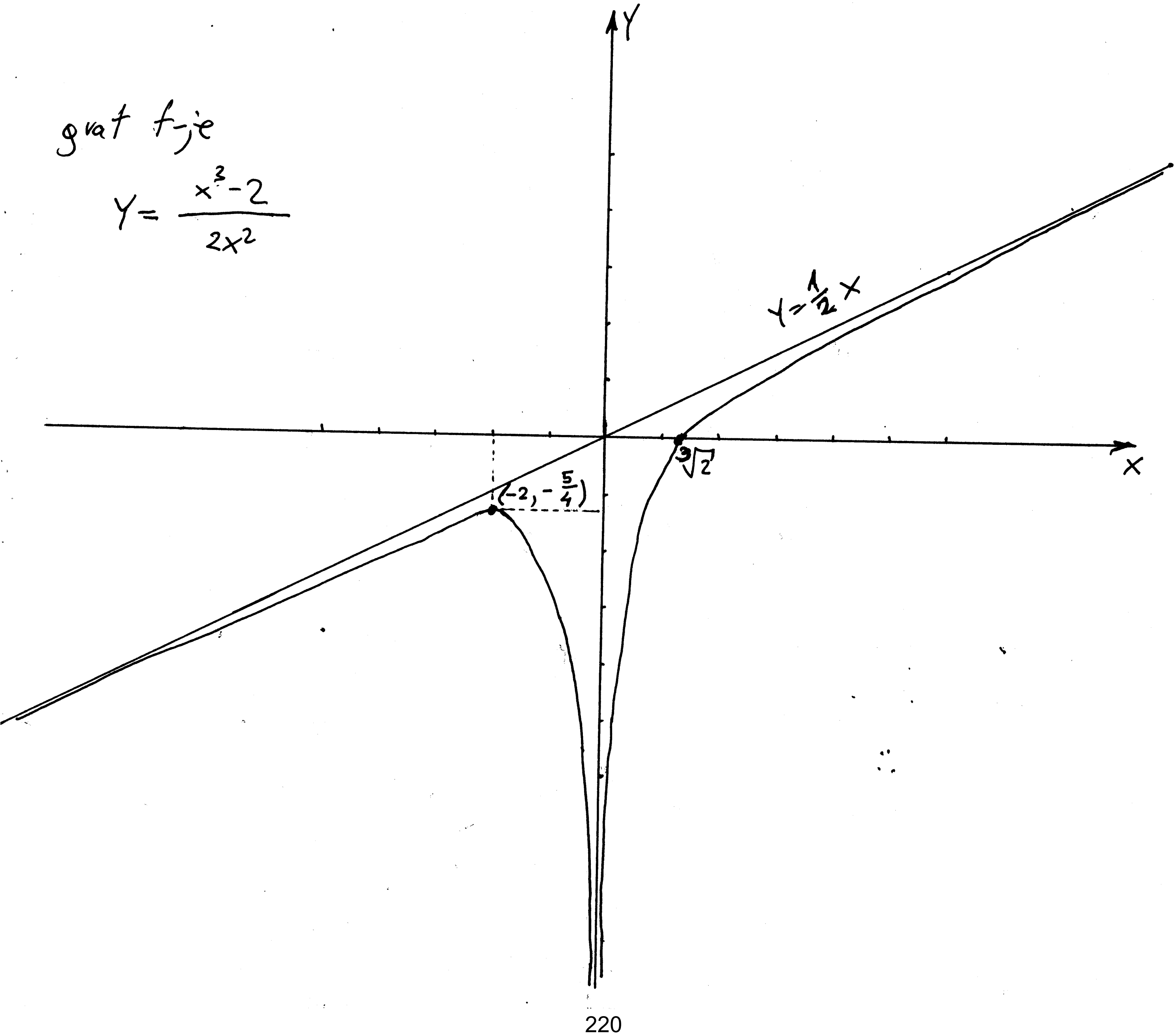
prevojne tačke i intervali konveksnosti i konkavnosti

$y'' = \left(\frac{x^3+8}{2x^3}\right)' = \frac{3x^2 \cdot 2x^3 - (x^3+8)\cdot 6x^2}{4x^6} = \frac{6x^5-6x^5-48}{4x^6} = \frac{-48}{4x^6} = -\frac{12}{x^6} < 0$

F-ja nema prevojnih tački i uvijek je negativna što znači uvijek je $\cap$ oblika.

graf f-je

$y = \frac{x^3-2}{2x^2}$



ⓘ Ispitati f-ju i nacrtati njen grafik $y = e^{\frac{x}{1-x}} - 1$.

Rj. definiciono područje

$1-x \neq 0$

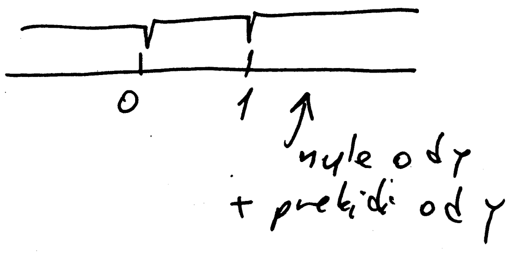
$x \neq 1$ $\quad \mathcal{D}: x \in (-\infty, 1) \cup (1, +\infty)$

parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$

f-ja nije ni parna ni neparna

f-ja nije periodična



nule, presjek sa y-osom, znak f-je

$y=0$ ako $e^{\frac{x}{1-x}} = 1$

tj $\frac{x}{1-x} = 0 \Rightarrow x=0$

(0,0) je nula f-je i presjek sa y-osom

$y > 0 \Leftrightarrow e^{\frac{x}{1-x}} - 1 > 0$

	$(-\infty, 0)$	$(0,1)$	$(1,+\infty)$
x	−	+	+
$1-x$	+	+	−
y	−	+	−

znak f-je

$e^{\frac{x}{1-x}} > 1$

$e^{\frac{x}{1-x}} > e^0$

$\frac{x}{1-x} > 0$

Ponašanje na krajevima intervala definisanosti i asimptote

za $x=1$ f-ja ima prekid

$$\lim_{x\to 1-0} f(x) = \lim_{x\to 1-0}\left(e^{\frac{x}{1-x}} - 1\right) = e^{\frac{1-0}{1-1+0}} - 1 = e^{\frac{1-0}{+0}} - 1 = e^{\infty} - 1 = \infty$$

$$\lim_{x\to 1+0} f(x) = \lim_{x\to 1+0}\left(e^{\frac{x}{1-x}} - 1\right) = e^{\frac{1+0}{1-1-0}} - 1 = e^{\frac{1+0}{-0}} - 1 = e^{-\infty} - 1 = \frac{1}{e^{\infty}} - 1 = -1$$

$x = 1$ je vertikalna asimptota (sa lijeve strane)

$$\lim_{x\to\mp\infty} f(x) = \lim_{x\to\mp\infty}\left(e^{\frac{x\,:x}{1-x\,:x}} - 1\right) =$$

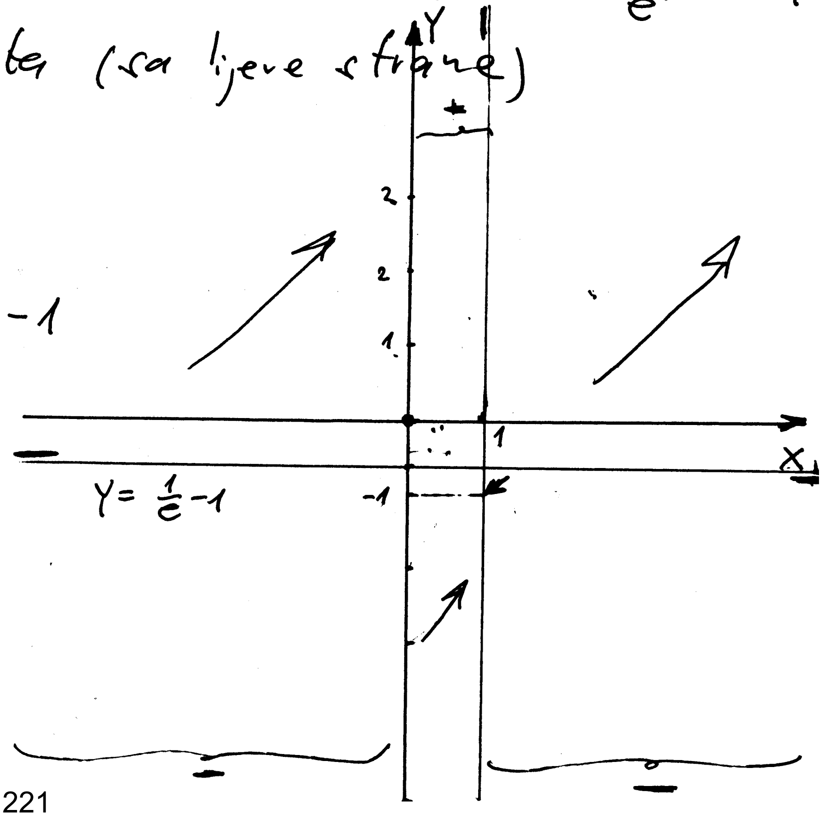
$$= \lim_{x\to\mp\infty}\left(e^{\frac{1}{\frac{1}{x} - 1}} - 1\right) = e^{-1} - 1 = \frac{1}{e} - 1$$

$y = \frac{1}{e} - 1 \approx -0{,}63$

je H. A.

kose asimptote nema

Poslije ovog koraka počinjemo sa skiciranjem grafa f-je



rast i opadanje

$$Y'=\left(e^{\frac{x}{1-x}}-1\right)'=e^{\frac{x}{1-x}}\cdot\left(\frac{x}{1-x}\right)'=\frac{1\cdot(1-x)-x\cdot(-1)}{(1-x)^2}e^{\frac{x}{1-x}}=\frac{e^{\frac{x}{1-x}}}{(1-x)^2}$$

$Y'=\frac{1}{(1-x)^2}e^{\frac{x}{1-x}}$ $Y'>0$ za $\forall x\in D$, f-ja ↗ za $\forall x$

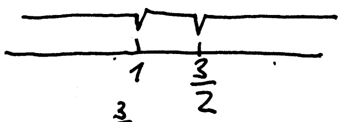
ekstremi f-je

$Y'\neq 0$ $\forall x$ f-ja nema ekstrema

$$Y''=\left(\frac{1}{(1-x)^2}e^{\frac{x}{1-x}}\right)'=(-2)(1-x)^{-3}(-1)e^{\frac{x}{1-x}}+\frac{1}{(1-x)^2}\cdot\frac{1}{(1-x)^2}e^{\frac{x}{1-x}}$$

$$Y''=\frac{2\cdot(1-x)+1}{(1-x)^4}e^{\frac{x}{1-x}}=\frac{-2x+3}{(1-x)^4}e^{\frac{x}{1-x}}$$ $Y''=0$ akko $x=\frac{3}{2}$

prekidi od Y + nule Y'' →



x	$(-\infty,1)$	$(1,\frac{3}{2})$	$(\frac{3}{2},+\infty)$
Y''	+	+	−
Y	∪	∪	∩

konveksnost i konkavnost

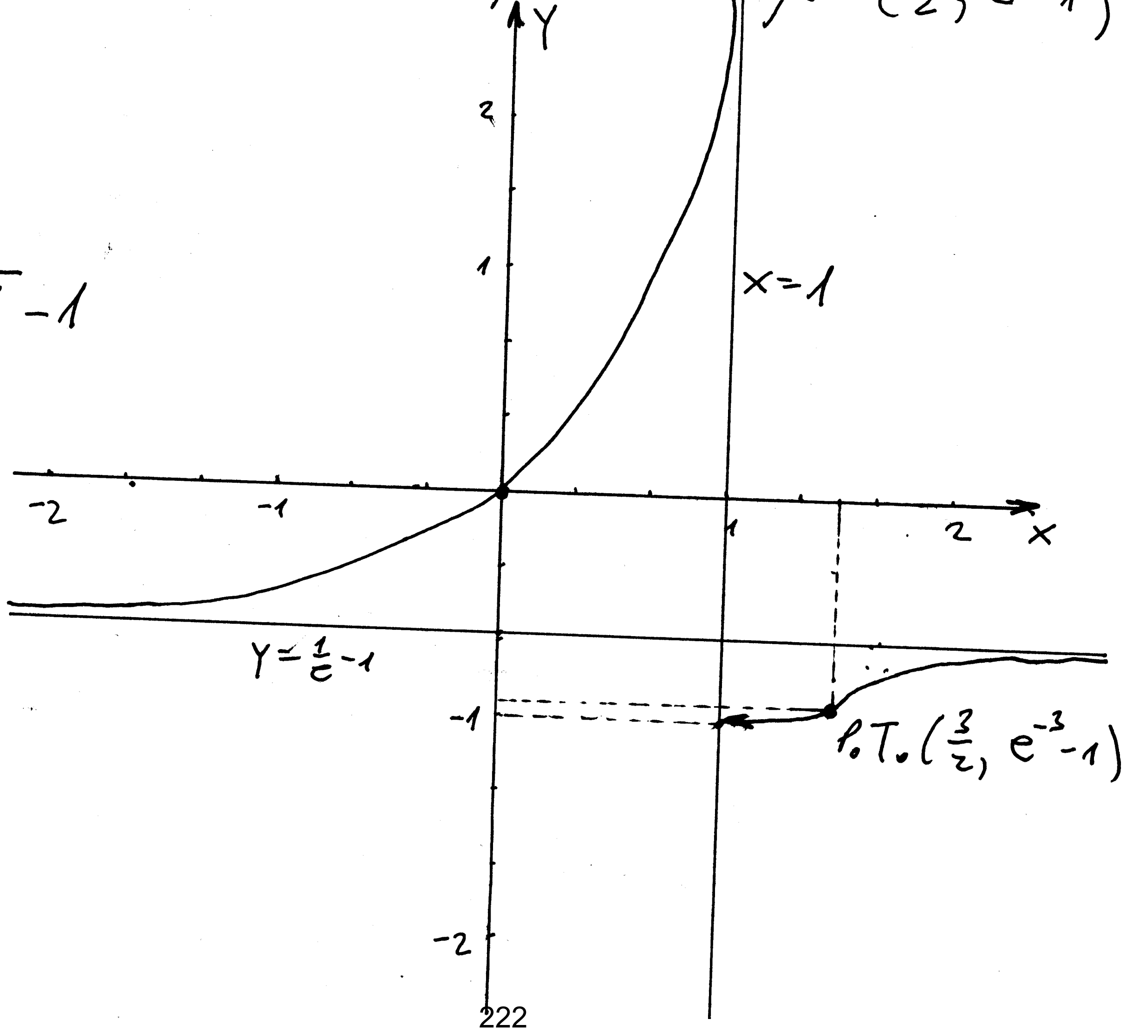
P.T.

$$f\left(\frac{3}{2}\right)=e^{\frac{\frac{3}{2}}{1-\frac{3}{2}}}-1=e^{\frac{\frac{3}{\cancel{2}}}{\frac{-1}{\cancel{2}}}}-1$$

$$f\left(\frac{3}{2}\right)=e^{-3}-1\approx-0,95$$

Prevojna tačka je $(\frac{3}{2},e^{-3}-1)$

grafik f-je
$y=e^{\frac{x}{1-x}}-1$



⑦ Ispitati f-ju i nacrtati njen grafik: $y = \frac{\ln^2 x + 1}{x^2}$.

R.: definiciono područje

$x \neq 0$ i $x > 0$

$\mathcal{D}: x \in (0, +\infty)$

parnost (neparnost), periodičnost

. $\mathcal{D}$ nije simetrično

$\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y = 0$ akko $\ln^2 x + 1 = 0$

$(\ln x)^2 = -1$

f-ja nema nulu

$f(0)$ nije definisano

f-ja ne siječe y-osu

$\ln^2 x + 1 > 0 \ \forall x \in \mathcal{D}$

$x^2 > 0 \ \forall x \in \mathcal{D}$

f-ja je uvijek pozitivna

ponašanje na krajevima intervala definisanosti i asimptote

za $x \leq 0$ f-ja nije definisana

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{\ln^2 x + 1}{x^2} = \frac{+\infty}{0+} = +\infty$$ $\Rightarrow$ $x = 0$ je vertikalna asimptota

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{\ln^2 x + 1}{x^2} \left(= \frac{\infty}{\infty}\right) \overset{L.P.}{=} \lim_{x \to +\infty} \frac{2\ln x \cdot \frac{1}{x}}{2x} = \lim_{x \to +\infty} \frac{\ln x}{x^2} \left(= \frac{\infty}{\infty}\right) \overset{L.P.}{=}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x}}{2x} = 0$$ $\Rightarrow$ $y = 0$ je horizontalna asimptota

f-ja nema kosu asimptotu

počinjemo skicirati grafik

rast i opadanje

$$y' = \left(\frac{\ln^2 x + 1}{x^2}\right)' = \frac{2\ln x \cdot \frac{1}{x} \cdot x^2 - (\ln^2 x + 1) 2x}{x^4}$$

$$= \frac{2x(\ln x - \ln^2 x - 1)}{x^4} = 2 \frac{\ln x - \ln^2 x - 1}{x^3}$$

$y' = 0$ akko $-\ln^2 x + \ln x - 1 = 0$

$\ln x = t$

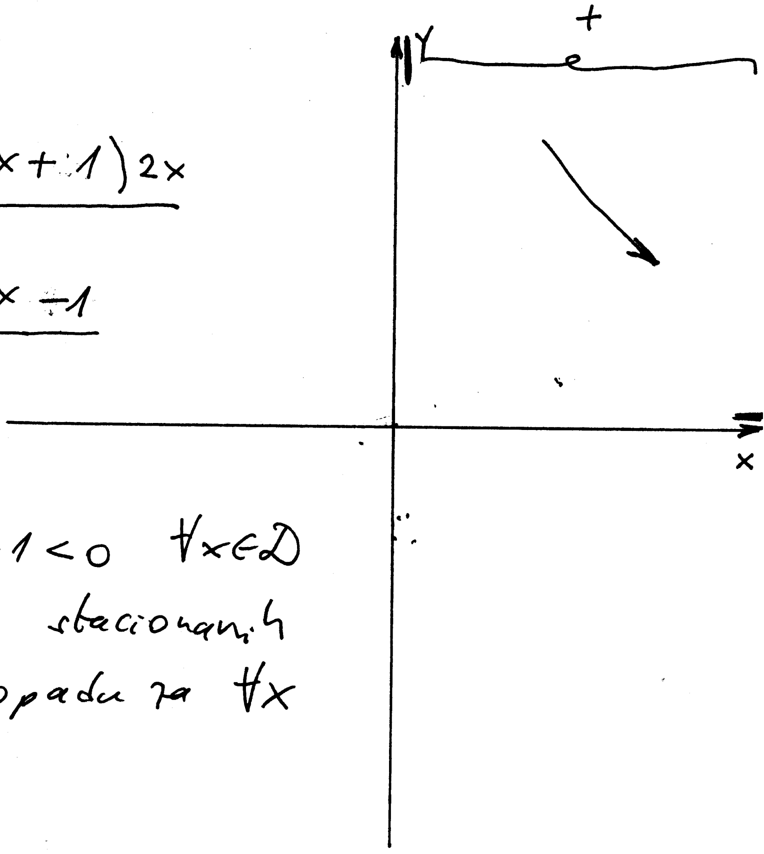
$-t^2 + t - 1 = 0$

$t^2 - t + 1 = 0$

$D = 1 - 4 < 0$

$-\ln^2 x + \ln x - 1 < 0 \ \forall x \in \mathcal{D}$

f-ja nema stacionarnih tački i opada za $\forall x$



ekstremi f-je

f-ja nema stacionarnih tački ⇒ f-ja nema ekstrema

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = 2\left(\frac{\ln x - \ln^2 x - 1}{x^3}\right)' = 2\,\frac{\left(\frac{1}{x} - 2\ln x\cdot\frac{1}{x}\right)x^{\cancel{3}} - (\ln x - \ln^2 x - 1)\cdot 3\cancel{x^2}}{x^{\cancel{6}4}} =$$

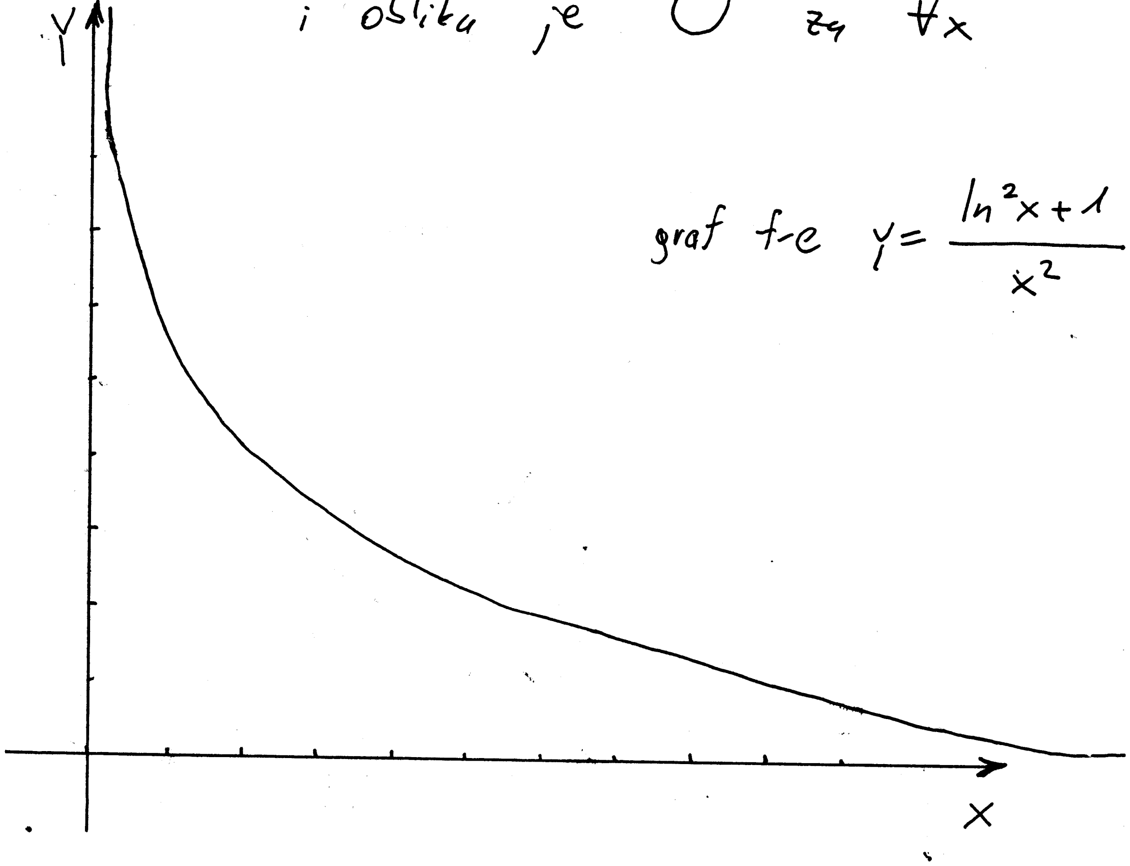
$$= 2\,\frac{1 - 2\ln x - 3\ln x + \underline{3\ln^2 x} + 3}{x^4} = 2\,\frac{3\ln^2 x - 5\ln x + 4}{x^4}$$

$3\ln^2 x - 5\ln x + 4 = 0$

$\ln x = t \qquad 3t^2 - 5t + 4 = 0$ $\Rightarrow$ $3\ln^2 x - 5\ln x + 4 > 0 \quad \forall x$

$D = 25 - 48 < 0 \qquad x^4 > 0 \quad \forall x$

$y'' > 0 \quad \forall x \in D$ $\Rightarrow$ f-ja nema prevojnih tački i oblika je $\cup$ za $\forall x$



graf f-e $y = \frac{\ln^2 x + 1}{x^2}$

(#) Ispitati f-ju i nacrtati joj grafik $$y=\frac{x^4-9x^2+12}{3x}.$$

R: definiciono područje

$\mathcal{D}: x \neq 0$

$x \in \mathbb{R}\setminus\{0\}$

parnost (neparnost), periodičnost

$$f(-x)=\frac{(-x)^4-9(-x)^2+12}{3(-x)}=-\frac{x^4-9x^2+12}{3x}=-f(x)$$

f-ja je neparna $\xrightarrow{\text{simetrična u odnosu na } (0,0)}$ (dovoljno ju je ispitati za $x>0$)

f-ja nije periodična

nule, presjek sa y-osom i znak f-je

$y=0$ akko $x^4-9x^2+12=0$

$x^2=t$

$t^2-9t+12=0$

$D=81-48=33$

$$t_{1,2}=\frac{9\pm\sqrt{33}}{2}$$

$x^2=\frac{9-\sqrt{33}}{2}$ $\qquad$ $x^2=\frac{9+\sqrt{33}}{2}$

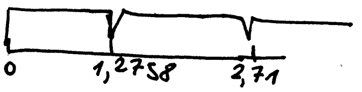
$x_1 \approx -1{,}2758$ $\qquad$ $x_2 \approx -2{,}7152$

$x_2 \approx 1{,}2758$ $\qquad$ $x_4 \approx 2{,}7152$

f(0) nije definisano

f-ja ne siječe Y-osu

znak f-je



← prekidi f-je i nule f-je

x	(0, 1,27)	(1,27, 3,71)	(2,71, +∞)
Y	+	−	+

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

Za $x=0$ f-ja ima prekid

$$\lim_{x\to 0+} f(x)=\lim_{x\to +0}\frac{x^4-9x^2+12}{3x}=\frac{12}{0+}=+\infty$$

$$\lim_{x\to 0-} f(x)=\lim_{x\to 0-}\frac{x^4-9x^2+12}{3x}=\frac{12}{0-}=-\infty$$

$\Rightarrow$ $x=0$ je V_0A_0

$$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\frac{x^4-9x^2+12 \;/:x}{3x \;/:x}=\lim_{x\to\pm\infty}\frac{x^3-9x+\frac{12}{x}}{3}=\pm\infty$$

$\Rightarrow$ f-ja nema H_0A_0

tražimo kosu asimptotu u obliku $y=kx+n$,

$$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x^4-9x^2+12 \;/:x^2}{3x^2 \;/:x^2}=\infty$$

f-ja nema kosu asimptotu

Nakon ovog koraka počinjemo skicirati graf f-je.

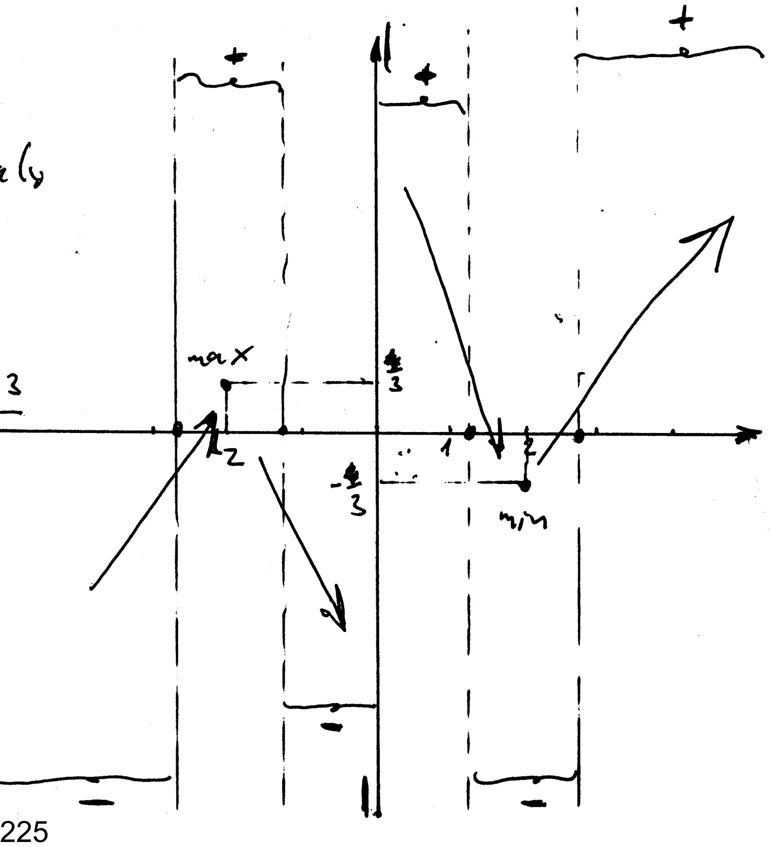
rast i opadanje

$$y'=\left(\frac{x^4-9x^2+12}{3x}\right)'=\frac{(4x^3-18x)3x-(x^4-9x^2+12)\cdot 3}{9x^2}$$

$$=\frac{12x^4-54x^2-3x^4+27x^2+36}{9x^2}=$$

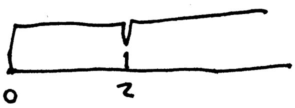
$$=\frac{9x^4-27x^2-36}{9x^2}=\frac{x^4-3x^2-4}{x^2}$$

$$y'=x^2-3-\frac{4}{x^2}$$



$y'=0$ akko $x^4-3x^2-4=0$

$t=x^2$



← prekidi f-je y + nule f-je y'

$t^2-3t-4=0$

$D=9+16=25$

$t_{1,2}=\frac{3\pm5}{2}$

$t_1=-1 \quad t_2=4$

$\Downarrow$

$x^2=4$

$x_1=-2 \quad x_2=2$

x	$(0,2)$	$(2,+\infty)$
y'	$-$	$+$
y	$\searrow$	$\nearrow$

min

$f(2)=\frac{16-36+12}{6}$

$f(2)=-\frac{8}{6}=-\frac{4}{3}$

ekstremi f-je

Na osnovu tabele rasta i opadanja i simetričnosti grafa f-ja ima minimum u $(2,-\frac{4}{3})$ i maksimum u $(-2,\frac{4}{3})$.

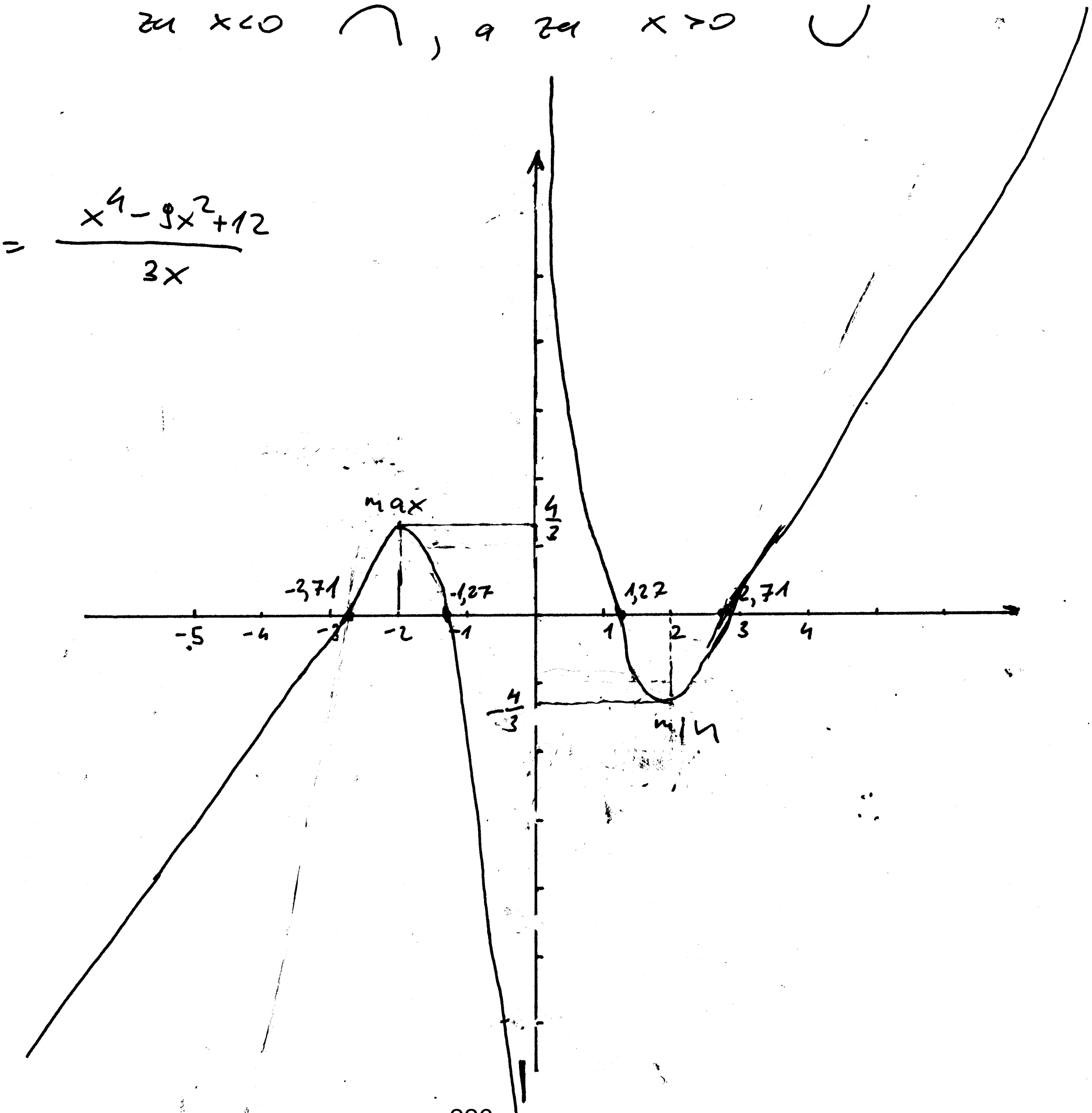
prevojne tačke i intervali konveksnosti i konkavnosti

$y''=\left(x^2-3-\frac{4}{x^2}\right)'=2x-4(-2)x^{-3}=2x+\frac{8}{x^3}$

$y''=\frac{2x^4+8}{x^3}$ kako je $2x^4+8>0 \ \forall x \Rightarrow$ f-ja nema prevojnih tački

za $x<0$ $\cap$, a za $x>0$ $\cup$

f-ja $y=\frac{x^4-9x^2+12}{3x}$



(#) Ispitati f-ju $y=\frac{ax+b}{x^2+x+1}$ i nacrtati joj grafik ako se zna da ona ima ekstrem u tački $T(1,\frac{2}{3})$.

Rj. $f(x)=\frac{ax+b}{x^2+x+1}$

$f(1)=\frac{2}{3} \Rightarrow \frac{a+b}{3}=\frac{2}{3}$

$a+b=2$

$$y'=\frac{a(x^2+x+1)-(ax+b)(2x+1)}{(x^2+x+1)^2}$$

$$y'=\frac{a(x^2+x+1)-(2ax^2+ax+2bx+b)}{(x^2+x+1)^2}$$

$$y'=\frac{-ax^2-2bx+a-b}{(x^2+x+1)^2}$$

U stacionarnoj tački f-ja može imati ekstrem

$y'=0 \Rightarrow -ax^2-2bx+a-b=0$

$x=1$

$-a-2b+a-b=0$

$-3b=0$

$b=0,\ a=2$

$$y=\frac{2x}{x^2+x+1}$$

$$y'=\frac{-2x^2+2}{(x^2+x+1)^2}$$

$$y'=(-2)\,\frac{x^2-1}{(x^2+x+1)^2}$$

definiciono područje

$x^2+x+1\neq 0$

f-ja je definisana za $\forall x$

nule, presjek sa x-osom, znak f-je

$y=0 \Rightarrow 2x=0 \Rightarrow x=0$

$(0,0)$ je presjek sa y-osom i nula f-je

Kako je $x^2+x+1>0\ \forall x$ to je

$y>0$ za $x>0$

$y<0$ za $x<0$ znak f-je

parnost (neparnost), periodičnost

$f(-x)=\frac{-2x}{x^2-x+1}$

f-ja nije ni parna ni neparna

f-ja nije periodična

ponašanje na krajevima intervala definisanosti i asimptote

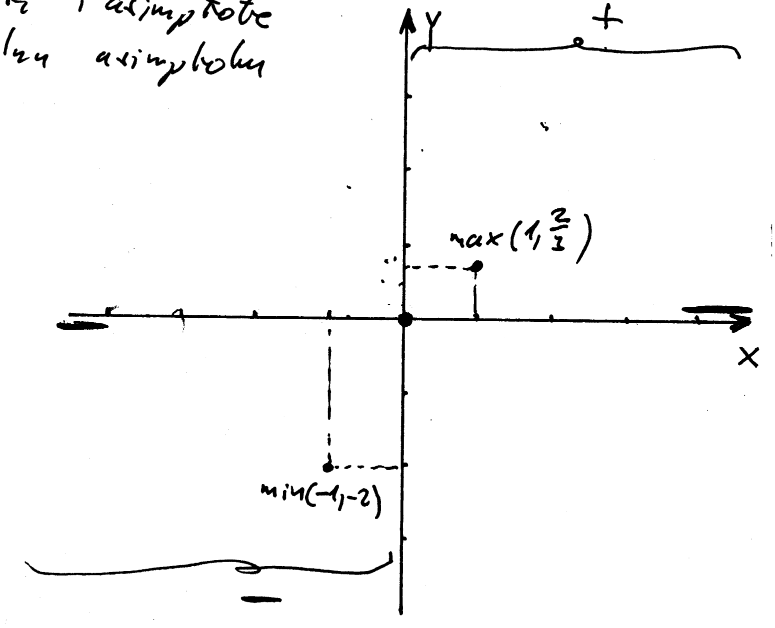
f-ja nema prekid $\Rightarrow$ f-ja nema vertikalnu asimptotu

$$\left.\begin{aligned}\lim_{x\to\infty} f(x)&=\lim_{x\to\infty}\frac{2x \ /:x}{x^2+x+1 \ /:x}=0\\ \lim_{x\to-\infty} f(x)&=\lim_{x\to-\infty}\frac{2x}{x^2+x+1}=0\end{aligned}\right\}\Rightarrow$$

$\Rightarrow x=0$ je H_0A_0

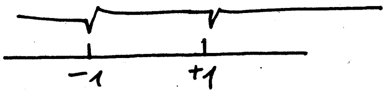
F-ja nema kosu asimptotu

Poslije ovog koraka počinjemo skicirati grafik f-je.



rast i opadanje

$$y'=(-2)\frac{x^2-1}{(x^2+x+1)^2}$$



x	$(-\infty,-1)$	$(-1,1)$	$(1,+\infty)$
y'	—	+	—
y	↘	↗	↘

min max

$$y'=0 \Rightarrow x=\pm 1$$

ekstremi f-je

$$f(-1)=\frac{-2}{1-1+1}=-2$$

F-ja ima minimum u tački $P(-1,-2)$
i maksimum u tački $(1,\frac{2}{3})$

$$f(1)=\frac{2}{1+1+1}=\frac{2}{3}$$

prevojne tačke i intervali konveksnosti i konkavnosti

$$y''=(-2)\left(\frac{x^2-1}{(x^2+x+1)^2}\right)'=(-2)\frac{2x(x^2+x+1)^{\overset{1}{\cancel{2}}}-(x^2-1)\,2\cancel{(x^2+x+1)}(2x+1)}{(x^2+x+1)^{\cancel{4}\,3}}$$

$$y''=(-2)\frac{2x^3+\cancel{2x^2}+2x-4x^3+4x-\cancel{2x^2}+2}{(x^2+x+1)^3}=(-2)\frac{-2x^3+x+2}{(x^2+x+1)^3}=(-2)\frac{(-2)(x^3-3x-1)}{(x^2+x+1)^3}$$

$$y''=4\frac{x^3-3x-1}{(x^2+x+1)^3}$$

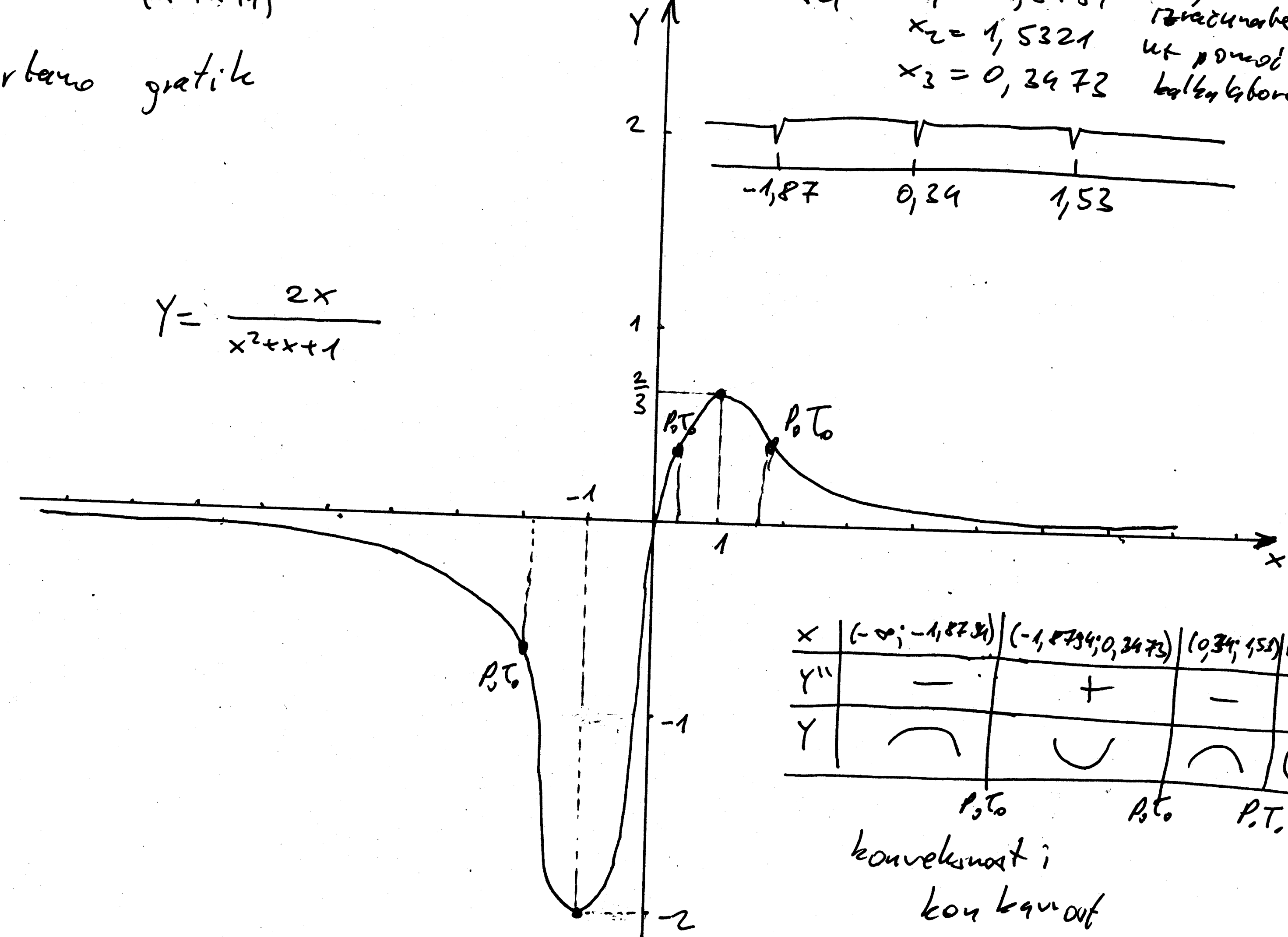
korjeni od x^3-3x-1 su

$x_1=-1,8734$
$x_2=1,5321$
$x_3=0,3473$

(vrijednosti izračunate uz pomoć kalkulatora)

crtamo grafik

$$y=\frac{2x}{x^2+x+1}$$



x	$(-\infty; -1,8734)$	$(-1,8734; 0,3473)$	$(0,34; 1,53)$	$(1,53;+\infty)$
y''	—	+	—	+
y	∩	∪	∩	∪

P_0T_0 P_0T_0 P_0T_0

konveksnost i konkavnost

(#) Ispitati f-ju i nacrtati joj grafik $Y = x e^{\frac{1}{2}(1-\frac{1}{x^2})}$

Rj. definiciono područje

$x \neq 0$

$\mathcal{D}: x \in \mathbb{R} \setminus \{0\}$

parnost (neparnost), periodičnost

$f(-x) = -x e^{\frac{1}{2}(1-\frac{1}{(-x)^2})} = -x e^{\frac{1}{2}(1-\frac{1}{x^2})} = -f(-x)$

f-ja je neparna
f-ja nije periodična

nule, presjek sa Y-osom, znak f-je

f(0) nije definisano

f-ja ne sječe Y-osu

$Y \neq 0, \ \forall x \in \mathcal{D}$

$(e^{\frac{1}{2}(1-\frac{1}{x^2})} > 0 \ \forall x)$

f-ja nema nulu

X	$(-\infty, 0)$	$(0, +\infty)$
Y	−	+

znak f-je

ponašanje na krajevima intervala definiranosti i asimptote

za x=0 f-ja ima prekid

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x e^{\frac{1}{2}(1-\frac{1}{x^2})} = (0-)\cdot e^{\frac{1}{2}(1-\infty)} = (0-) e^{-\infty} = \frac{0-}{e^{\infty}} = 0$$

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} x e^{\frac{1}{2}(1-\frac{1}{x^2})} = (0+) e^{-\infty} = 0$$

f-ja nema vertikalnu asimptotu

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} x e^{\frac{1}{2}(1-\frac{1}{x^2})} = (-\infty)\cdot e^{\frac{1}{2}} = -\infty$$

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} x e^{\frac{1}{2}(1-\frac{1}{x^2})} = \infty \cdot e^{\frac{1}{2}} = \infty$$

f-ja nema horizontalnu asimptotu

tražimo kosu asimptotu u obliku

$Y = kx + n$

$$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} e^{\frac{1}{2}(1-\overset{\nearrow 0}{\frac{1}{x^2}})} = e^{\frac{1}{2}} = \sqrt{e}$$

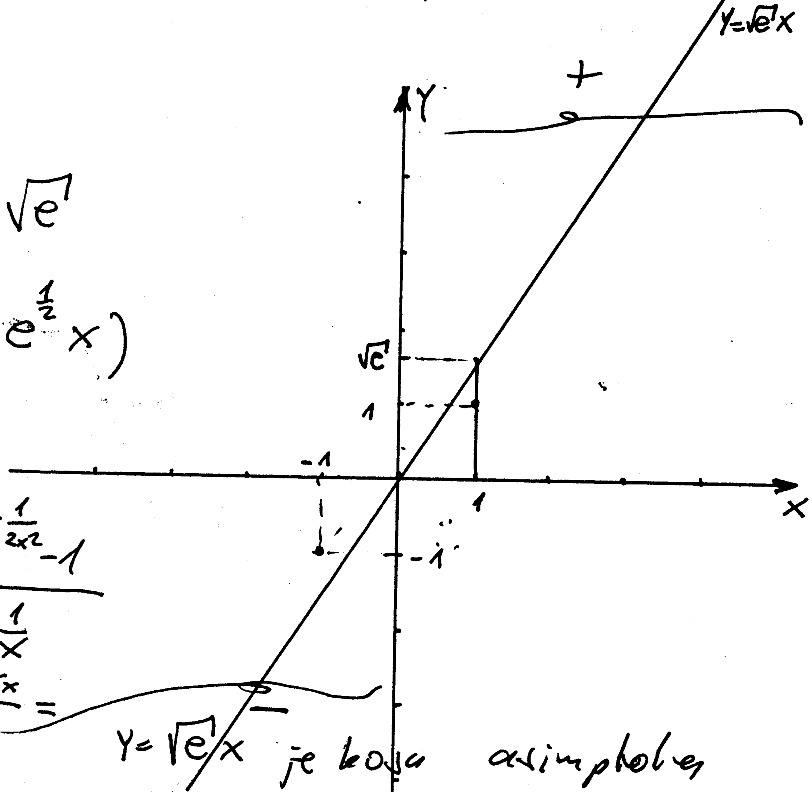
$$n = \lim_{x\to\infty} [f(x) - k\cdot x] = \lim_{x\to\infty} (x e^{\frac{1}{2}(1-\frac{1}{x^2})} - e^{\frac{1}{2}} x)$$

$$= \lim_{x\to\infty} x(e^{\frac{1}{2}(1-\frac{1}{x^2})} - e^{\frac{1}{2}}) =$$

$$= \lim_{x\to\infty} e^{\frac{1}{2}} x (e^{\frac{-1}{2x^2}} - 1) = \sqrt{e} \lim_{x\to\infty} \frac{e^{-\frac{1}{2x^2}} - 1}{\frac{1}{x}}$$

$$\left(= \frac{0}{0}\right) \overset{L.P.}{=} \sqrt{e} \lim_{x\to\infty} \frac{e^{-\frac{1}{2x^2}} \cdot (-\frac{1}{2})(-2) \frac{1}{\cancel{x^3}} \overset{\frac{1}{x}}{}}{\frac{-1}{\cancel{x^2}}} =$$

$$= \sqrt{e} \lim_{x\to\infty} \frac{-e^{-\frac{1}{2x^2}}}{x} = \sqrt{e} \cdot \frac{-1}{\infty} = 0$$



$Y = \sqrt{e}\,x$ je kosa asimptota

počinjemo sa skiciranjem grafika $\sqrt{e} \approx 1{,}64$

rast i opadanje

$$\left(-\frac{1}{x^2}\right)' = (-x^{-2})' = 2x^{-3} = \frac{2}{x^3}$$

$$y' = \left(x e^{\frac{1}{2}(1-\frac{1}{x^2})}\right)' = e^{\frac{1}{2}(1-\frac{1}{x^2})} + x e^{\frac{1}{2}(1-\frac{1}{x^2})} \cdot \left(\frac{1}{2}\left(1-\frac{1}{x^2}\right)\right)' =$$

$$= e^{\frac{1}{2}(1-\frac{1}{x^2})} + x e^{\frac{1}{2}(1-\frac{1}{x^2})} \cdot \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{x^3} = e^{\frac{1}{2}(1-\frac{1}{x^2})}\left(1+\frac{1}{x^2}\right)$$

$y' = 0$ akko $1 + \frac{1}{x^2} = 0$

$\frac{x^2+1}{x^2} = 0$

$y' > 0 \ \forall x \Rightarrow$ f-ja uvijek raste

f-ja nema ekstrema

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left[e^{\frac{1}{2}(1-\frac{1}{x^2})}\left(1+\frac{1}{x^2}\right)\right]' = e^{\frac{1}{2}(1-\frac{1}{x^2})} \cdot \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{x^3}\left(1+\frac{1}{x^2}\right) + e^{\frac{1}{2}(1-\frac{1}{x^2})} \cdot \frac{-2}{x^3} =$$

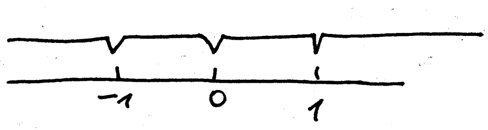
$$= e^{\frac{1}{2}(1-\frac{1}{x^2})}\left(\frac{1}{x^3} + \frac{1}{x^5} - \frac{2}{x^3}\right) = \left(\frac{1}{x^5} - \frac{1}{x^3}\right) e^{\frac{1}{2}(1-\frac{1}{x^2})}$$

$f(1) = 1 e^{\frac{1}{2}0} = 1$

$y'' = 0$ akko $\frac{1-x^2}{x^5} = 0 \Rightarrow 1 - x^2 = 0$

$x = \pm 1$

prekidi od $y \to$
+ nule od y''



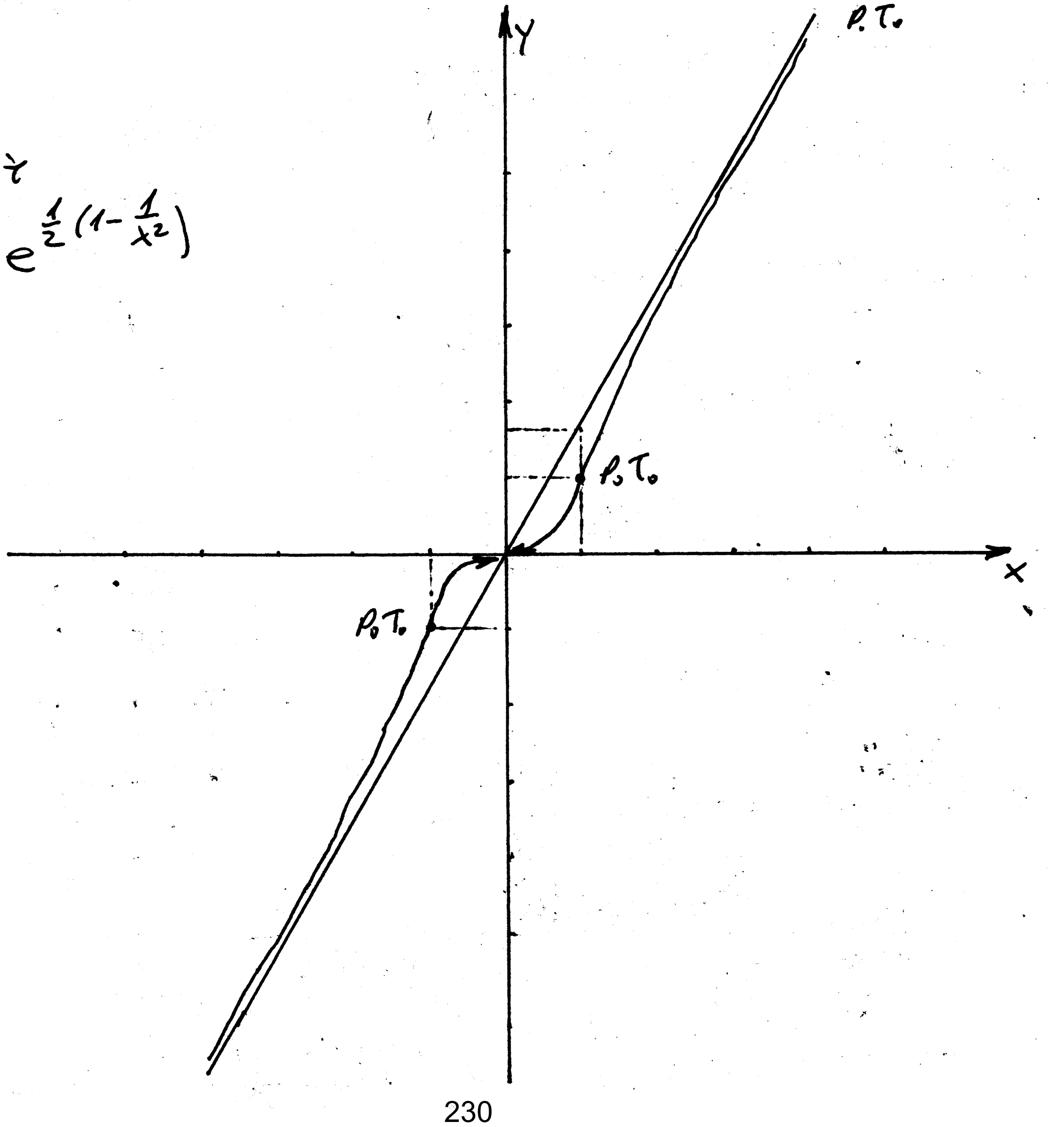
	$(0, 1)$	$(1, +\infty)$
y''	+	−
y	∪	∩

P.T.

$(1, 1)$ i $(-1, -1)$ su prevojne tačke

graf. f-je

$y = x e^{\frac{1}{2}(1-\frac{1}{x^2})}$



(#) Ispitati f-ju i nacrtati joj grafik $y=\ln\frac{x^2-3x+2}{x^2+1}$.

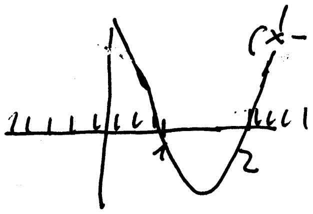
Rj: definiciono područje

Kako je $x^2+1>0\ \forall x\in\mathbb{R}$

to iz $\frac{x^2-3x+2}{x^2+1}>0 \Rightarrow$

treba da bude $x^2-3x+2>0$

$(x-1)(x-2)>0$



$\mathcal{D}: x\in(-\infty,1)\cup(2,+\infty)$

nule, presjek sa y-osom, znak

$y=0 \Rightarrow \ln\frac{x^2-3x+2}{x^2+1}=0$

$\Rightarrow \frac{x^2-3x+2}{x^2+1}=1 \quad /\cdot x^2+1$

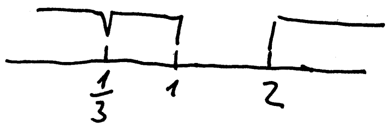
$\cancel{x^2}-3x+2=\cancel{x^2}+1$

$3x=1 \Rightarrow x=\frac{1}{3}$

$(\frac{1}{3},0)$ je nula f-je

$y(0)=\ln 2\approx 0{,}6931$

$(0,\ln 2)$ je presjek sa y-osom



← prekidi y + nule y

x	$(-\infty,\frac{1}{3})$	$(\frac{1}{3},1)$	$(2,+\infty)$
y	+	−	−

znak f-je

parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

ponašanje na krajevima intervala definisanosti i asimptote

f-ja ima prekid za $x=1$ i $x=2$

$\lim\limits_{x\to1-0} f(x)=\lim\limits_{x\to1-0}\ln\frac{x^2-3x+2}{x^2+1}=\ln\frac{(1-0)^2-3(1-0)+2}{(1-0)^2+1}=\ln(0_+)=-\infty \Rightarrow$

$\Rightarrow x=1$ je V_0A_0 (sa lijeve str.)

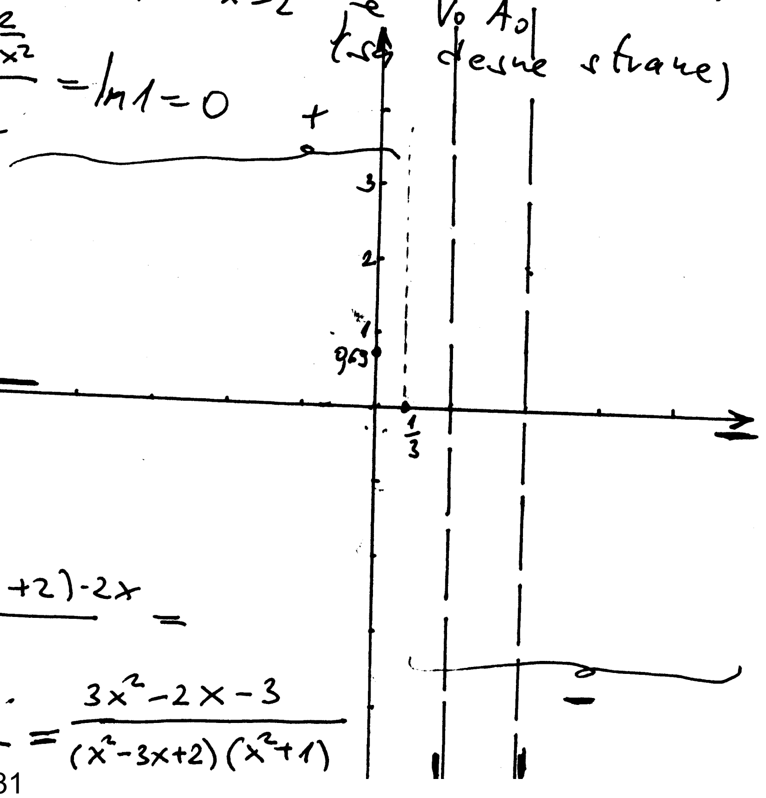
$\lim\limits_{x\to2+0} f(x)=\lim\limits_{x\to2+0}\ln\frac{x^2-3x+2}{x^2+1}=\ln(0+)=-\infty \Rightarrow x=2$ je V_0A_0 (sa desne strane)

$\lim\limits_{x\to\mp\infty} f(x)=\lim\limits_{x\to\mp\infty}\ln\frac{x^2-3x+2}{x^2+1}=\lim\limits_{x\to\mp\infty}\frac{1-\frac{3}{x}+\frac{2}{x^2}}{1+\frac{1}{x^2}}=\ln 1=0$

$\Rightarrow y=0$ je H_0A_0

K_0A_0 nema.

počinjemo sa skiciranjem grafa



rast i opadanje

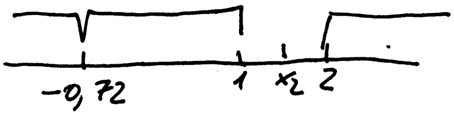
$y'=\frac{x^2+1}{x^2-3x+2}\cdot\left(\frac{x^2-3x+2}{x^2+1}\right)'$

$y'=\frac{\cancel{x^2+1}}{x^2-3x+2}\cdot\frac{(2x-3)(x^2+1)-(x^2-3x+2)\cdot 2x}{(x^2+1)^{\cancel{2}}}=$

$=\frac{\cancel{2x^3}+2x-3x^2-3-\cancel{2x^3}+6x^2-4x}{(x^2-3x+2)(x^2+1)}=\frac{3x^2-2x-3}{(x^2-3x+2)(x^2+1)}$

$Y'=0 \Rightarrow 3x^2-2x-3=0 \Rightarrow x_{1,2}=\frac{2\mp\sqrt{4+36}}{6}$

$x_{1,2}=\frac{2\pm2\sqrt{10}}{6}=\frac{1\pm\sqrt{10}}{3}$



$x_1=\frac{1+\sqrt{10}}{3}\approx1,387\notin D$

$x_2=\frac{1-\sqrt{10}}{3}\approx-0,721\in D$

x	$(-\infty, \frac{1-\sqrt{10}}{3})$	$(\frac{1-\sqrt{10}}{3}, 1)$	$(2, +\infty)$
Y'	+	−	+
Y	↗	↘	↗
	max		

ekstremi f-je

$f(\frac{1-\sqrt{10}}{3})\approx1,016$

F-ja ima maksimum u tački (-0,72; 1,02)

prevojne tačke i intervali konveksnosti i konkavnosti

$Y''=\left(\frac{3x^2-2x-3}{(x^2-3x+2)(x^2+1)}\right)'=$ ZA VJEŽBU ... $=\frac{-6x^5+15x^4-30x^2+30x-13}{(x^2-3x+2)^2(x^2+1)^2}$

$Y''=0$ akko $x=-1,5166$ (izračunato uz pomoć kalkulatora)

Kako je brojnik u y'' previše složen nije potrebno praviti tabelu konveksnosti i konkavnosti

grafik f-je

$Y=\ln\frac{x^2-3x+2}{x^2+1}$